Lecture 1

§1.1 Solutions of Linear Equations and Inequalities in One Variable

1. Functions

Definition 1.1. A **function** f is a special relation between x and y such that each input x results in at most one output y.
The symbol $f(x)$ is read "f of x" and is called the **value of f at x**.

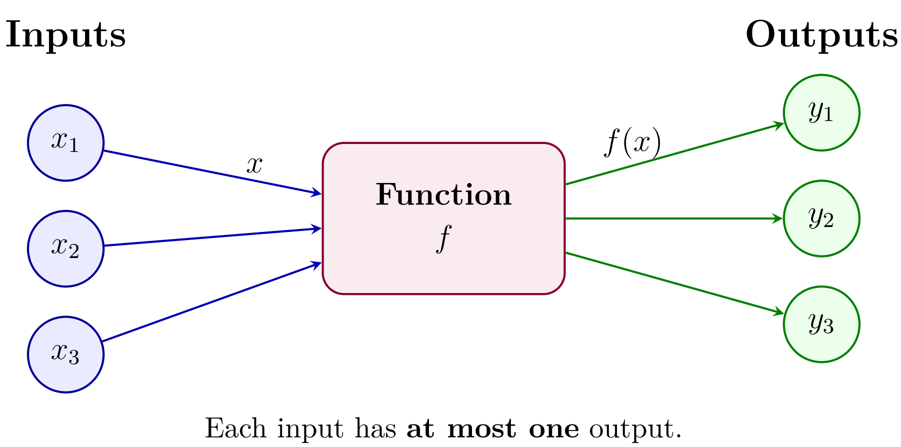


Figure 1: A function assigns each input x to at most one output $f(x)$.

Evaluating a Function

Example. Let

$$f(x) = 4x - 1.$$

Evaluate

$$f(1), \qquad f\left(\frac{1}{2}\right), \qquad f(-2), \qquad f(0), \qquad f(x), \qquad f(f(x)).$$

To evaluate a function, substitute the given input for x.
For $f(1)$,

$$\begin{aligned} f(1) &= 4(1) - 1 \\ &= 3. \end{aligned}$$

For $f\left(\frac{1}{2}\right)$,

$$\begin{aligned} f\left(\frac{1}{2}\right) &= 4\left(\frac{1}{2}\right) - 1 \\ &= 2 - 1 \\ &= 1. \end{aligned}$$

For $f(-2)$,

$$\begin{aligned} f(-2) &= 4(-2) - 1 \\ &= -8 - 1 \\ &= -9. \end{aligned}$$

For $f(0)$,

$$\begin{aligned} f(0) &= 4(0) - 1 \\ &= -1. \end{aligned}$$

For the input x, we simply have

$$\boxed{f(x) = 4x - 1}.$$

Now consider $f(f(x))$.
Since

$$f(x) = 4x - 1,$$

we substitute $4x - 1$ into the function:

$$\begin{aligned} f(f(x)) &= f(4x - 1) \\ &= 4(4x - 1) - 1 \\ &= 16x - 4 - 1 \\ &= 16x - 5. \end{aligned}$$

Therefore,

$$\boxed{f(f(x)) = 16x - 5}.$$

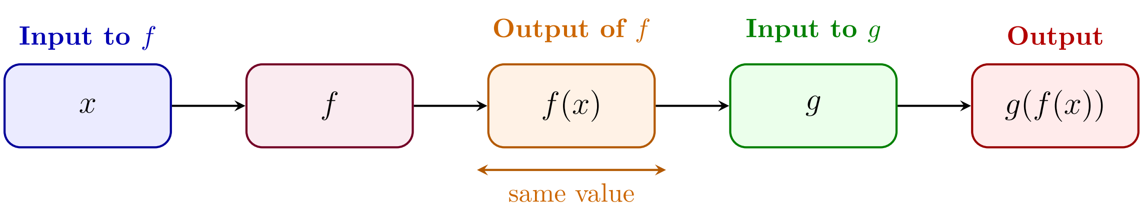
1.1 Composite Functions

Let f and g be functions of x.
The composite function g of f, denoted $g \circ f$, is defined by

$$\boxed{(g \circ f)(x) = g(f(x))}.$$

Similarly, the composite function f of g, denoted $f \circ g$, is defined by

$$\boxed{(f \circ g)(x) = f(g(x))}.$$



$$x \xrightarrow{f} f(x) \xrightarrow{g} g(f(x))$$

Figure 2: In the composition $g \circ f$, x is the input to f. The output $f(x)$ then becomes the input to g.

Example. Let

$$f(x) = 4x - 1, \qquad g(x) = x - 1.$$

Find
$$(g \circ f)(x) \qquad \text{and} \qquad (f \circ g)(x).$$

Step 1: Find $(g \circ f)(x)$
By definition,
$$(g \circ f)(x) = g(f(x)).$$

Since
$$f(x) = 4x - 1,$$
substitute $4x - 1$ into g:
$$\begin{aligned}(g \circ f)(x) &= g(4x - 1) \\ &= (4x - 1) - 1 \\ &= 4x - 2.\end{aligned}$$

Therefore,
$$\boxed{(g \circ f)(x) = 4x - 2}.$$

Step 2: Find $(f \circ g)(x)$
By definition,
$$(f \circ g)(x) = f(g(x)).$$

Since
$$g(x) = x - 1,$$
substitute $x - 1$ into f:
$$\begin{aligned}(f \circ g)(x) &= f(x - 1) \\ &= 4(x - 1) - 1 \\ &= 4x - 4 - 1 \\ &= 4x - 5.\end{aligned}$$

Therefore,
$$\boxed{(f \circ g)(x) = 4x - 5}.$$

Remark. In general,
$$(g \circ f)(x) \neq (f \circ g)(x).$$

For this example,
$$4x - 2 \neq 4x - 5.$$

Thus, composition of functions is generally **not commutative**.

1.2 Operations with Functions

Let f and g be functions of x. Define
$$\begin{aligned}(f + g)(x) &= f(x) + g(x), \\ (f - g)(x) &= f(x) - g(x), \\ (f \cdot g)(x) &= f(x)g(x), \\ \left(\frac{f}{g}\right)(x) &= \frac{f(x)}{g(x)}, \qquad g(x) \neq 0.\end{aligned}$$

2. Expressions, Equations, and Solutions

Definition 2.1. An **expression** is a meaningful string of numbers, variables, and mathematical operations.

For example,

$$3x - 2.$$

An expression represents a quantity, but it does not state that two quantities are equal.

Definition 2.2. An **equation** is a statement that two quantities or algebraic expressions are equal.

For example,

$$3x - 2 = 7.$$

Definition 2.3. A **solution** is a value of the variable that makes the equation true.

For example, $x = 3$ is a solution of

$$3x - 2 = 7$$

because

$$3(3) - 2 = 7,$$

so

$$9 - 2 = 7,$$

and therefore

$$7 = 7.$$

Definition 2.4. A **solution set** is the set of all possible solutions of an equation.

For example,

$$3x - 2 = 7$$

has only the solution

$$x = 3.$$

On the other hand,

$$2(x - 1) = 2x - 2$$

is true for every real number x. Its solution set is

$$\boxed{\mathbb{R}}.$$

3. Properties of Equality

Substitution Property

The equation formed by substituting one expression for an equal expression is equivalent to the original equation.

For example,

$$3(x - 3) - \frac{1}{2}(4x - 18) = 4.$$

First distribute:

$$3x - 9 - (2x - 9) = 4.$$

Thus,

$$3x - 9 - 2x + 9 = 4,$$
$$x = 4.$$

Therefore,

$$\boxed{x = 4}.$$

3.1 Addition Property

The equation formed by adding the same quantity to both sides of an equation is equivalent to the original equation.
For example,

$$x - 4 = 6.$$

Add 4 to both sides:

$$x - 4 + 4 = 6 + 4.$$

The 4's cancel:

$$x = 10.$$

Therefore,

$$\boxed{x = 10}.$$

Similarly,

$$x + 5 = 12.$$

Add -5 to both sides:

$$x + 5 + (-5) = 12 + (-5).$$

The 5's cancel:

$$x = 7.$$

Therefore,

$$\boxed{x = 7}.$$

3.2 Multiplication Property

The equation formed by multiplying both sides of an equation by the same nonzero quantity is equivalent to the original equation.
For example,

$$\frac{1}{3}x = 6.$$

Multiply both sides by 3:

$$3\left(\frac{1}{3}x\right) = 3(6).$$

The factors of 3 cancel:

$$x = 18.$$

Therefore,

$$\boxed{x = 18}.$$

Similarly,

$$5x = 20.$$

Divide both sides by 5:

$$\frac{5x}{5} = \frac{20}{5}.$$

Therefore,

$$x = 4.$$

Hence,

$$\boxed{x = 4}.$$

Remark. The same nonzero quantity must be applied to both sides of an equation. Division by zero is undefined.

4. Solving Linear Equations

Using the properties of equality above, we can solve any linear equation in one variable.

General Procedure

A useful procedure is:

1. Eliminate fractions by multiplying every term by the least common denominator.
2. Remove parentheses using the distributive property.
3. Combine like terms.
4. Collect variable terms on one side.
5. Isolate the variable.
6. Verify the solution by substitution.

Example 1

Solve

$$\frac{3x}{4} + 3 = \frac{x-1}{3}.$$

Step 1: Eliminate fractions.
Multiply both sides by

$$12.$$

$$12\left(\frac{3x}{4} + 3\right) = 12\left(\frac{x-1}{3}\right).$$

Thus,

$$9x + 36 = 4x - 4.$$

Step 2: Collect the variable terms.
Subtract

$$4x$$

from both sides:

$$9x + 36 - 4x = 4x - 4 - 4x.$$

Therefore,

$$5x + 36 = -4.$$

Step 3: Collect the constants.
Subtract

$$36$$

from both sides:

$$5x + 36 - 36 = -4 - 36.$$

Thus,

$$5x = -40.$$

Step 4: Isolate the variable.
Divide both sides by

$$5.$$

$$\frac{5x}{5} = \frac{-40}{5}.$$

Therefore,

$$\boxed{x = -8}.$$

Step 5: Verify.
Substitute

$$x = -8$$

into the original equation:

$$\frac{3(-8)}{4} + 3 = -6 + 3 = -3,$$

while

$$\frac{-8 - 1}{3} = \frac{-9}{3} = -3.$$

Hence,

$$-3 = -3.$$

Therefore,

$$\boxed{x = -8}$$

is verified.

Solving a Formula for a Variable

Example. Solve

$$-2x + 6y = 4$$

for y.
We want to isolate y.
Step 1: Move the x-term.
Add $2x$ to both sides:

$$-2x + 6y + 2x = 4 + 2x.$$

Thus,

$$6y = 2x + 4.$$

Step 2: Isolate y.
Divide both sides by 6:

$$\frac{6y}{6} = \frac{2x + 4}{6}.$$

Therefore,

$$y = \frac{2x+4}{6}.$$

Factor 2 from the numerator:

$$y = \frac{2(x+2)}{2(3)}.$$

Cancel the common factor:

$$\boxed{y = \frac{x+2}{3}}.$$

5. Applications of Linear Equations

5.1 Profit

Suppose the relationship between a firm's profit P and the number of items sold x is

$$5x - 4P = 1200.$$

(a) How many units must be produced and sold for the firm to make a profit of \$150?
Substitute $P = 150$:

$$5x - 4(150) = 1200.$$

Simplify:

$$5x - 600 = 1200.$$

Step 1: Move the constant term.
Add 600 to both sides:

$$5x - 600 + 600 = 1200 + 600.$$

Thus,

$$5x = 1800.$$

Step 2: Isolate x.
Divide by 5:

$$\frac{5x}{5} = \frac{1800}{5}.$$

Therefore,

$$\boxed{x = 360}.$$

The firm must sell

$$\boxed{360 \text{ units}}.$$

(b) Solve the equation for P in terms of x.
Start with

$$5x - 4P = 1200.$$

Step 1: Move the x-term.
Subtract $5x$ from both sides:

$$5x - 4P - 5x = 1200 - 5x.$$

Thus,

$$-4P = 1200 - 5x.$$

Step 2: Isolate P.
Divide both sides by -4:

$$\frac{-4P}{-4} = \frac{1200 - 5x}{-4}.$$

Therefore,

$$P = \frac{5x - 1200}{4}.$$

Hence,

$$\boxed{P = \frac{5x - 1200}{4}}.$$

When 240 units are sold, substitute $x = 240$:

$$P = \frac{5(240) - 1200}{4}.$$

Thus,

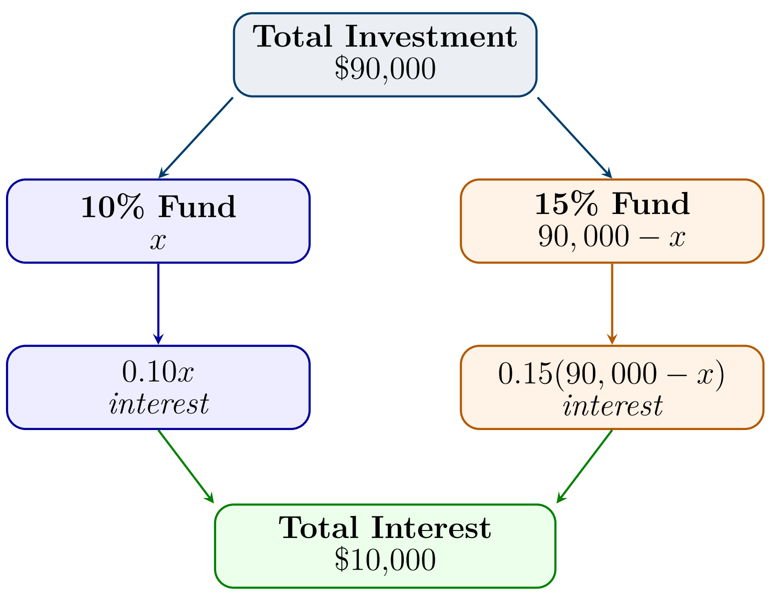
$$P = \frac{1200 - 1200}{4} = \frac{0}{4} = 0.$$

Therefore,

$$\boxed{P = 0}.$$

5.2 Investment

Jill Ball has $\$90,000$ to invest. She has chosen one relatively safe investment fund with an annual yield of 10% and another riskier fund with a 15% annual yield.
How much should she invest in each fund if she wants to earn $\$10,000$ in one year?



$$0.10x + 0.15(90,000 - x) = 10,000$$

Figure 3: The $90,000 is divided between two funds whose interest must total $10,000.

Let x be the amount invested at 10%.

Then

$$90,000 - x$$

is invested at 15%.
The interest equation is

$$0.10x + 0.15(90,000 - x) = 10,000.$$

Distribute:

$$0.10x + 13,500 - 0.15x = 10,000.$$

Combine like terms:

$$-0.05x + 13,500 = 10,000.$$

Step 1: Move the constant term.
Subtract 13,500 from both sides:

$$-0.05x + 13,500 - 13,500 = 10,000 - 13,500.$$

Thus,

$$-0.05x = -3,500.$$

Step 2: Isolate x.
Divide by -0.05:

$$\frac{-0.05x}{-0.05} = \frac{-3,500}{-0.05}.$$

Therefore,

$$\boxed{x = 70,000}.$$

So

$$\boxed{\$70,000}$$

should be invested at 10%.
The remaining amount is

$$90,000 - 70,000 = 20,000.$$

Therefore,

$$\boxed{\$20,000}$$

should be invested at 15%.
Check:

$$0.10(70,000) + 0.15(20,000) = 7,000 + 3,000 = 10,000.$$

The investment amounts are verified.

5.3 Percent Change

A woman making \$2,000 per month has her salary reduced by 10%. One year later she receives a 20% raise over her reduced salary.
Find her salary after the raise and the percent change from the original salary.
Step 1: Find the 10% reduction.

$$2000(0.10) = 200.$$

Her reduced salary is

$$2000 - 200 = 1800.$$

Thus,

$$\boxed{\$1,800}$$

is her reduced salary.

Step 2: Find the 20% raise.

$$1800(0.20) = 360.$$

Her new salary is

$$1800 + 360 = 2160.$$

Therefore,

$$\boxed{\$2,160}$$

is her salary after the raise.

Step 3: Find the change from the original salary.

$$2160 - 2000 = 160.$$

The percent change is

$$\frac{160}{2000}(100\%) = 8\%.$$

Therefore, her final salary represents an

$$\boxed{8\% \text{ increase}}$$

from the original salary.

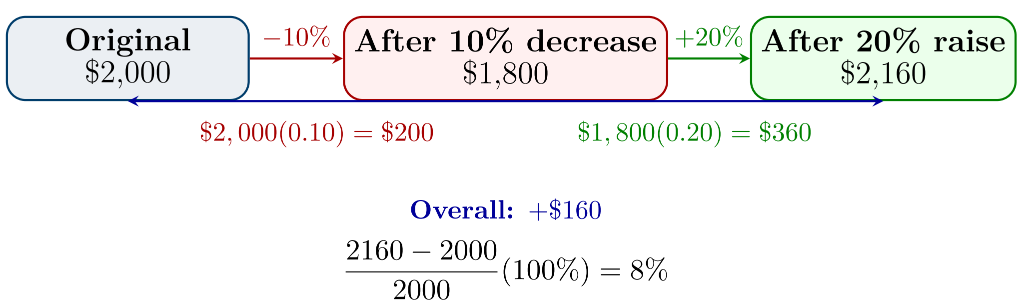


Figure 4: A 10% decrease followed by a 20% increase results in an overall 8% increase.

6. Inequalities

Definition 6.1. An **inequality** is a statement that one quantity is greater than or less than another quantity.

The common inequality symbols are

$$<, \qquad >, \qquad \leq, \qquad \geq .$$

6.1 Substitution Property

The inequality formed by substituting one expression for an equal expression is equivalent to the original inequality.
For example,

$$5x - 4x + 2 < 6.$$

Combine like terms:

$$x + 2 < 6.$$

Subtract 2 from both sides:

$$x + 2 - 2 < 6 - 2.$$

Therefore,

$$\boxed{x < 4}.$$

The solution set is

$$\boxed{\{x : x < 4\}}.$$

6.2 Addition Property

The inequality formed by adding the same quantity to both sides of an inequality is equivalent to the original inequality.
For example,

$$x - 4 < 6.$$

Add 4 to both sides:

$$x - 4 + 4 < 6 + 4.$$

Thus,

$$\boxed{x < 10}.$$

Similarly,

$$x + 5 \geq 12.$$

Add -5 to both sides:

$$x + 5 + (-5) \geq 12 + (-5).$$

Therefore,

$$\boxed{x \geq 7}.$$

6.3 Multiplication Property

Multiplying both sides of an inequality by the same positive quantity does not change the direction of the inequality.
However, multiplying or dividing by a negative quantity reverses the direction.
For example,

$$\frac{1}{3}x > 6.$$

Multiply both sides by 3:

$$3\left(\frac{1}{3}x\right) > 3(6).$$

Thus,

$$\boxed{x > 18}.$$

Now consider

$$-x \leq 25.$$

Divide both sides by -1.
Because -1 is negative, reverse the inequality:

$$\frac{-x}{-1} \geq \frac{25}{-1}.$$

Therefore,

$$\boxed{x \geq -25}.$$

Remark. When multiplying or dividing an inequality by a negative number, the inequality symbol must be reversed.

Solving Linear Inequalities

Example. Solve

$$-x + 8 \leq 2x - 4.$$

Method 1: Gather the x-terms on the left.
Subtract $2x$ from both sides:

$$-x + 8 - 2x \leq 2x - 4 - 2x.$$

Thus,

$$-3x + 8 \leq -4.$$

Subtract 8:

$$-3x + 8 - 8 \leq -4 - 8.$$

Therefore,

$$-3x \leq -12.$$

Divide by -3.
Because we divide by a negative number, reverse the inequality:

$$\frac{-3x}{-3} \geq \frac{-12}{-3}.$$

Hence,

$$\boxed{x \geq 4}.$$

Method 2: Gather the x-terms on the right.
Start with

$$-x + 8 \leq 2x - 4.$$

Add x to both sides:

$$-x + 8 + x \leq 2x - 4 + x.$$

Thus,

$$8 \leq 3x - 4.$$

Add 4:

$$8 + 4 \leq 3x - 4 + 4.$$

Therefore,

$$12 \leq 3x.$$

Divide by 3:

$$\frac{12}{3} \leq \frac{3x}{3}.$$

Thus,

$$4 \leq x.$$

Therefore,

$$\boxed{x \geq 4}.$$

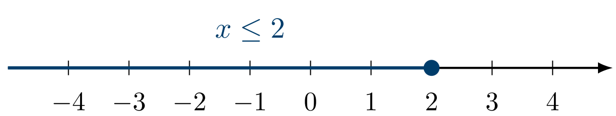
Both methods give the same solution:

$$\boxed{x \geq 4}.$$

Graphing Inequalities on a Number Line

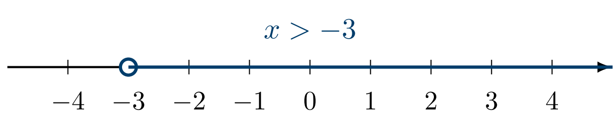
- Use an **open circle** when the endpoint is not included.
- Use a **closed circle** when the endpoint is included.

$x \leq 2$



The closed circle at 2 indicates that 2 is included in the solution set.

$x > -3$



The open circle at -3 indicates that -3 is not included in the solution set.

Practice Questions

Solve each problem.

1. Solve:

$$3x - 7 = 11.$$

2. Solve:

$$\frac{x+2}{3} = 5.$$

3. Solve:

$$4(x-2) = 2x + 6.$$

4. Solve:

$$-3x + 7 \geq 19.$$

5. Solve:
$$5 - 2x < 13.$$

6. Solve:
$$\frac{x-1}{4} + 2 = 5.$$

7. Solve for y:
$$3x + 2y = 12.$$

8. A company's profit is modeled by
$$P = 8x - 1200.$$
What is the profit when $x = 250$?

9. A person's salary is reduced by 15%, then increased by 15%. Is the final salary equal to the original salary? Explain.

10. Solve and graph:
$$-2x + 6 \leq 10.$$