Lecture 3

Section 1.3: Linear Functions

1. Linear Functions

Definition 1.1. A **linear function** is a function of the form

$$y = f(x) = mx + b,$$

where m and b are constants.

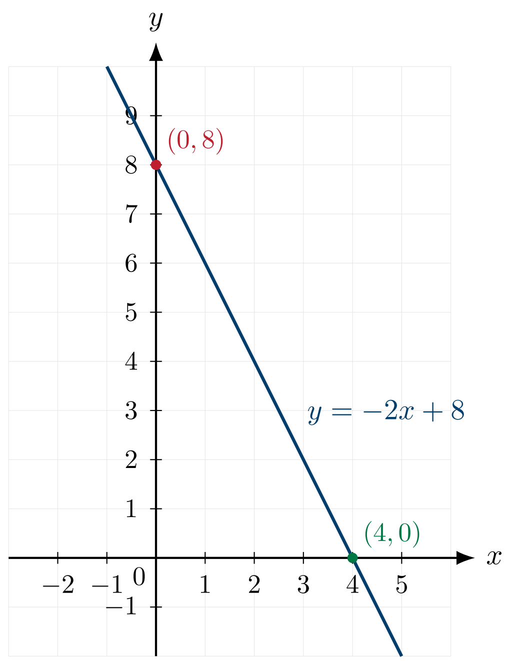
The constant m determines the slope of the line, while b determines the y-intercept.
For example,

$$y = -2x + 8$$

is a linear function with

$$\boxed{m = -2} \qquad \text{and} \qquad \boxed{b = 8}.$$

The graph of a linear function is a straight line.



A linear function can be uniquely determined using only two distinct points on its graph.

2. Intercepts

Definition 2.1. The point(s) where a graph intersects the coordinate axes are called the **intercepts**.

The x-**intercept** is the point where the graph crosses the x-axis.
Since every point on the x-axis has $y = 0$, find the x-intercept by setting

$$\boxed{y = 0}.$$

The y**-intercept** is the point where the graph crosses the y-axis.
Since every point on the y-axis has $x = 0$, find the y-intercept by setting

$$\boxed{x = 0}.$$

Remark. For an x-intercept, use $y = 0$. For a y-intercept, use $x = 0$.

2.1 Finding Intercepts from a Linear Equation

Example. Find the intercepts of

$$3x + 2y = 12.$$

Step 1: Find the x-intercept.
Set

$$y = 0.$$

Then

$$3x + 2(0) = 12.$$

Thus,

$$3x = 12,$$

so

$$x = 4.$$

Therefore,

$$\boxed{(4, 0)}$$

is the x-intercept.
Step 2: Find the y-intercept.
Set

$$x = 0.$$

Then

$$3(0) + 2y = 12.$$

Thus,

$$2y = 12,$$

so

$$y = 6.$$

Therefore,

$$\boxed{(0, 6)}$$

is the y-intercept.

Example: Find the intercepts of

$$5x - 8y = 60.$$

Step 1: Find the x-intercept.
Set

$$y = 0.$$

Then

$$5x - 8(0) = 60.$$

Therefore,

$$5x = 60.$$

Divide by 5:

$$x = 12.$$

Hence,

$$\boxed{(12, 0)}$$

is the x-intercept.

Step 2: Find the y-intercept.

Set

$$x = 0.$$

Then

$$5(0) - 8y = 60.$$

Thus,

$$-8y = 60.$$

Divide by -8:

$$y = -\frac{60}{8} = -\frac{15}{2}.$$

Hence,

$$\boxed{\left(0, -\frac{15}{2}\right)}$$

is the y-intercept.

Answer:

$$\boxed{x\text{-intercept} = (12, 0)}$$

and

$$\boxed{y\text{-intercept} = \left(0, -\frac{15}{2}\right)}.$$

Remark. When graphing a line, the two intercepts provide two convenient points. Plot the points and draw the line through them.

Example. Find the intercepts of

$$x = 4y.$$

Step 1: Find the x-intercept.

Set

$$y = 0.$$

Then

$$x = 4(0) = 0.$$

Thus,

$$\boxed{(0, 0)}$$

is the x-intercept.

Step 2: Find the y-intercept.

Set

$$x = 0.$$

Then

$$0 = 4y,$$

so

$$y = 0.$$

Therefore,

$$\boxed{(0, 0)}$$

is also the y-intercept.
Thus, both intercepts occur at the origin.

3. Slope

Definition 3.1. If a nonvertical line passes through the points

$$P_1(x_1, y_1) \qquad \text{and} \qquad P_2(x_2, y_2),$$

its **slope**, denoted by m, is

$$\boxed{m = \frac{y_2 - y_1}{x_2 - x_1}}.$$

The numerator

$$\Delta y = y_2 - y_1$$

represents the change in y, while

$$\Delta x = x_2 - x_1$$

represents the change in x.
Thus,

$$\boxed{m = \frac{\Delta y}{\Delta x}}.$$

Remark. The slope of a vertical line is undefined because its change in x is zero.

3.1 Finding the Slope from Two Points

Example. Find the slope of the line passing through

$$(-2, 1) \qquad \text{and} \qquad (5, 3).$$

Use

$$m = \frac{y_2 - y_1}{x_2 - x_1}.$$

Substitute:

$$m = \frac{3 - 1}{5 - (-2)}.$$

Therefore,

$$m = \frac{2}{7}.$$

Hence,

$$\boxed{m = \frac{2}{7}}.$$

Interpretation:
For every increase of 7 units in x, the value of y increases by 2 units.

Example. Find the slope of the line passing through

$$(25, 28) \qquad \text{and} \qquad (18, -14).$$

Use

$$m = \frac{y_2 - y_1}{x_2 - x_1}.$$

Let

$$(x_1, y_1) = (25, 28), \qquad (x_2, y_2) = (18, -14).$$

Then

$$m = \frac{-14 - 28}{18 - 25}.$$

Simplify:

$$m = \frac{-42}{-7}.$$

Therefore,

$$\boxed{m = 6}.$$

Remark. The order of the points does not matter as long as the order is the same in the numerator and denominator.

Example. Find the slope of the line passing through

$$(3, -3) \qquad \text{and} \qquad (-1, 3).$$

Using

$$m = \frac{y_2 - y_1}{x_2 - x_1},$$

we obtain

$$m = \frac{3 - (-3)}{-1 - 3}.$$

Thus,

$$m = \frac{6}{-4} = -\frac{3}{2}.$$

Therefore,

$$\boxed{m = -\frac{3}{2}}.$$

3.2 Zero and Undefined Slope

There are two important special cases.
Horizontal line:
If the y-coordinates are equal, then

$$\Delta y = 0.$$

Therefore,

$$m = 0.$$

For example, the slope through

$$(2, 1) \qquad \text{and} \qquad (-5, 1)$$

is

$$m = \frac{1-1}{-5-2} = \frac{0}{-7} = \boxed{0}.$$

Vertical line:
If the x-coordinates are equal, then

$$\Delta x = 0.$$

Division by zero is undefined.
For example, the slope through

$$(4, 2) \qquad \text{and} \qquad (4, -3)$$

is

$$m = \frac{-3-2}{4-4} = \frac{-5}{0}.$$

Therefore,

$$\boxed{\text{undefined}}.$$

Remark. Remember:

$$\boxed{\text{horizontal} \Rightarrow m = 0}$$

and

$$\boxed{\text{vertical} \Rightarrow m \text{ is undefined}}.$$

4. Parallel and Perpendicular Lines

Two distinct nonvertical lines are **parallel** if and only if their slopes are equal.
Thus,

$$\boxed{m_1 = m_2}.$$

Two distinct nonvertical lines are **perpendicular** if and only if their slopes are negative reciprocals.
If one line has slope $m \neq 0$, then a perpendicular line has slope

$$\boxed{-\frac{1}{m}}.$$

For example, if

$$m_1 = 3,$$

then a perpendicular line has slope

$$\boxed{-\frac{1}{3}}.$$

If

$$m_1 = -\frac{2}{5},$$

then a perpendicular line has slope

$$\boxed{\frac{5}{2}}.$$

5. Point-Slope Form

Definition 5.1. The equation of a line passing through the point

$$(x_1, y_1)$$

with slope m can be written in **point-slope form**:

$$\boxed{y - y_1 = m(x - x_1)}.$$

5.1 Using Point-Slope Form

Example. Find the equation of the line passing through

$$(-3, 4)$$

with slope

$$m = \frac{1}{4}.$$

Use

$$y - y_1 = m(x - x_1).$$

Substitute:

$$y - 4 = \frac{1}{4}\left(x - (-3)\right).$$

Thus,

$$y - 4 = \frac{1}{4}(x + 3).$$

Add 4:

$$y = \frac{1}{4}(x + 3) + 4.$$

Distribute:

$$y = \frac{1}{4}x + \frac{3}{4} + 4.$$

Therefore,

$$\boxed{y = \frac{1}{4}x + \frac{19}{4}}.$$

Example: Find the equation of the line passing through $(-1, 3)$ with slope $m = -\frac{5}{4}$.

Step 1: Write point-slope form.

$$y - y_1 = m(x - x_1).$$

Substitute

$$x_1 = -1, \qquad y_1 = 3, \qquad m = -\frac{5}{4}:$$

$$y - 3 = -\frac{5}{4}\left(x - (-1)\right).$$

Thus,

$$y - 3 = -\frac{5}{4}(x + 1).$$

Step 2: Distribute.

$$y - 3 = -\frac{5}{4}x - \frac{5}{4}.$$

Step 3: Add 3.

$$y = -\frac{5}{4}x - \frac{5}{4} + 3.$$

Since

$$3 = \frac{12}{4},$$

we obtain

$$y = -\frac{5}{4}x + \frac{7}{4}.$$

Therefore,

$$\boxed{y = -\frac{5}{4}x + \frac{7}{4}}.$$

5.2 Horizontal Lines

A horizontal line has slope

$$\boxed{m = 0}.$$

Example. Find the equation of the line passing through

$$(-2, 1)$$

with slope 0.
Using point-slope form,

$$y - 1 = 0(x + 2).$$

Therefore,

$$\boxed{y = 1}.$$

5.3 Vertical Lines

A vertical line has an undefined slope.
Example. Find the equation of the line passing through

$$(-2, 1)$$

with undefined slope.
Every point on a vertical line has the same x-coordinate.
Therefore,

$$\boxed{x = -2}.$$

Example: Find the equation of the line passing through $(-2, 2)$ with undefined slope.
Since the line is vertical, use its constant x-coordinate:

$$\boxed{x = -2}.$$

6. Slope-Intercept Form

Definition 6.1. The **slope-intercept form** of the equation of a line with slope m and y-intercept b is

$$\boxed{y = mx + b}.$$

In this form,

$$\boxed{m = \text{slope}}$$

and

$$\boxed{b = y\text{-intercept}}.$$

6.1 Writing an Equation from Slope and Intercept

Example. Write the equation of the line with slope

$$m = \frac{1}{4}$$

and y-intercept

$$b = -3.$$

Start with

$$y = mx + b.$$

Substitute

$$m = \frac{1}{4}, \qquad b = -3:$$

$$y = \frac{1}{4}x - 3.$$

Therefore,

$$\boxed{y = \frac{1}{4}x - 3}.$$

6.2 Homework-Type Example

Example. Write the slope-intercept equation of the line with slope

$$m = -4$$

and y-intercept

$$b = \frac{5}{2}.$$

Start with

$$y = mx + b.$$

Substitute:

$$y = (-4)x + \frac{5}{2}.$$

Therefore,

$$\boxed{y = -4x + \frac{5}{2}}.$$

6.3 Converting to Slope-Intercept Form

Example. Write

$$3x + 2y = 12$$

in slope-intercept form.
Solve for y.
Subtract $3x$:

$$2y = 12 - 3x.$$

Divide by 2:

$$y = \frac{12 - 3x}{2}.$$

Separate the terms:

$$y = 6 - \frac{3}{2}x.$$

Reorder:

$$\boxed{y = -\frac{3}{2}x + 6}.$$

Thus,

$$\boxed{m = -\frac{3}{2}} \qquad \text{and} \qquad \boxed{b = 6}.$$

7. Finding the Equation of a Line from Two Points

When two points are given, use the following procedure:

1. Find the slope using
$$m = \frac{y_2 - y_1}{x_2 - x_1}.$$
2. Use either point in point-slope form.
3. Simplify if slope-intercept form is requested.

7.1 Homework-Type Example

Example. Find the equation of the line passing through

$$(4, 5) \qquad \text{and} \qquad (-1, -5).$$

Step 1: Find the slope.

$$m = \frac{-5 - 5}{-1 - 4}.$$

Thus,

$$m = \frac{-10}{-5} = 2.$$

Therefore,

$$\boxed{m = 2}.$$

Step 2: Use point-slope form.
Using $(4, 5)$,

$$y - 5 = 2(x - 4).$$

Step 3: Simplify.
Distribute:

$$y - 5 = 2x - 8.$$

Add 5:

$$y = 2x - 3.$$

Therefore,

$$\boxed{y = 2x - 3}.$$

7.2 Another Homework-Type Example

Example. Find the equation of the line passing through

$$(8, 4) \qquad \text{and} \qquad (-3, 3).$$

Step 1: Find the slope.

$$m = \frac{3 - 4}{-3 - 8} = \frac{-1}{-11} = \frac{1}{11}.$$

Thus,

$$\boxed{m = \frac{1}{11}}.$$

Step 2: Use point-slope form.
Using $(8, 4)$,

$$y - 4 = \frac{1}{11}(x - 8).$$

Step 3: Simplify.

$$y - 4 = \frac{x}{11} - \frac{8}{11}.$$

Add 4:

$$y = \frac{x}{11} - \frac{8}{11} + 4.$$

Since

$$4 = \frac{44}{11},$$

we obtain

$$\boxed{y = \frac{1}{11}x + \frac{36}{11}}.$$

8. Applications of Linear Functions

8.1 Population Model

The population of U.S. males, y, in thousands, projected from 2015 to 2060 can be modeled by

$$y = 1125.9x + 142,960,$$

where x is the number of years after 2000.

(a) Find the slope and y-intercept.

Compare with

$$y = mx + b.$$

Therefore,

$$\boxed{m = 1125.9}$$

and

$$\boxed{b = 142,960}.$$

(b) Interpret the y-intercept.
When

$$x = 0,$$

the year is 2000.
Therefore, the model predicts

$$142,960$$

thousand males, or

$$\boxed{142.96 \text{ million}}$$

in 2000.
(c) Interpret the slope.
The slope is

$$1125.9$$

thousand people per year.
Thus, the population is predicted to increase by approximately

$$\boxed{1.1259 \text{ million people per year}}.$$

8.2 Homework-Type Population Model

Suppose that from 2020 to 2060, the number of females in the U.S. under the age of 18, in millions, is modeled by

$$N = 0.150x + 38.4,$$

where x is the number of years after 2020.
(a) Find the slope.
Compare with

$$N = mx + b.$$

Therefore,

$$\boxed{m = 0.150}.$$

(b) Interpret the slope.
The slope represents the change in population for each one-year increase in x.
Thus,

$$\boxed{\text{the population increases by 0.150 million females per year}}.$$

Equivalently,

$$0.150 \text{ million} = 150,000.$$

So the model predicts an increase of

$$\boxed{150,000 \text{ females per year}}.$$

(c) Predict the population in 2030.
Since x is the number of years after 2020,

$$x = 2030 - 2020 = 10.$$

Substitute

$$x = 10$$

into the model:

$$N = 0.150(10) + 38.4.$$

Therefore,

$$N = 1.5 + 38.4 = 39.9.$$

Hence,

$$\boxed{N = 39.9 \text{ million}}.$$

Remark. Always convert the calendar year into x before evaluating the model. Here, 2030 corresponds to $x = 10$, not $x = 2030$.

8.3 Sleep and Age

Each day, a young person should sleep 8 hours plus $\frac{1}{4}$ hour for each year the person is under 18 years of age.
Assuming that the relation is linear, write the equation relating hours of sleep y and age x.
The number of years under 18 is

$$18 - x.$$

Thus,

$$y = 8 + \frac{1}{4}(18 - x).$$

Distribute:

$$y = 8 + \frac{18}{4} - \frac{1}{4}x.$$

Therefore,

$$\boxed{y = -\frac{1}{4}x + \frac{25}{2}}.$$

The slope is

$$\boxed{-\frac{1}{4}},$$

so the recommended sleep decreases by

$$\boxed{\frac{1}{4} \text{ hour}}$$

for each additional year of age.

8.4 Linear Cost Model

An electric utility company charges a base fee of \$17.38 per month plus 7.32 cents per kilowatt-hour.

Let x be the number of kilowatt-hours used and let y be the monthly charge in dollars.
Step 1: Convert cents to dollars.
Since

$$100 \text{ cents} = \$1,$$

we have

$$7.32 \text{ cents} = \boxed{\$0.0732}.$$

Step 2: Identify the slope and intercept.
The charge for each kilowatt-hour is the slope:

$$\boxed{m = 0.0732}.$$

The fixed monthly charge is the y-intercept:

$$\boxed{b = 17.38}.$$

Step 3: Write the equation.
Using

$$y = mx + b,$$

we obtain

$$\boxed{y = 0.0732x + 17.38}.$$

Remark. The coefficient of x represents the cost per kilowatt-hour, while the constant term represents the fixed monthly charge.

9. Forms of Linear Equations

A line can be represented in several useful forms.

9.1 General Form

$$\boxed{Ax + By = C}$$

where A, B, and C are constants.

9.2 Point-Slope Form

$$\boxed{y - y_1 = m(x - x_1)}$$

where (x_1, y_1) is a point on the line and m is the slope.

9.3 Slope-Intercept Form

$$\boxed{y = mx + b}$$

where m is the slope and b is the y-intercept.

9.4 Vertical Line

$$\boxed{x = a}$$

A vertical line has an undefined slope.

9.5 Horizontal Line

$$\boxed{y = b}$$

A horizontal line has slope

$$\boxed{0}.$$

10. Choosing the Correct Form

The information given in a problem usually suggests which form should be used.

Given Information	**Useful Form**
Slope and y-intercept	$y = mx + b$
One point and slope	$y - y_1 = m(x - x_1)$
Two points	$m = \dfrac{y_2 - y_1}{x_2 - x_1}$, then point-slope
Vertical line	$x = a$
Horizontal line	$y = b$
Intercepts	Set $x = 0$ or $y = 0$

11. Finding the Equation of a Line

Example. Find the equation of the line passing through

$$(-2, 1) \qquad \text{and} \qquad (5, 3).$$

Step 1: Find the slope.

$$m = \frac{3 - 1}{5 - (-2)} = \frac{2}{7}.$$

Thus,

$$\boxed{m = \frac{2}{7}}.$$

Step 2: Use point-slope form.
Using $(-2, 1)$,

$$y - 1 = \frac{2}{7}(x + 2).$$

Step 3: Simplify.

$$y - 1 = \frac{2}{7}x + \frac{4}{7}.$$

Add 1:

$$y = \frac{2}{7}x + \frac{4}{7} + 1.$$

Therefore,

$$\boxed{y = \frac{2}{7}x + \frac{11}{7}}.$$