Lecture 4

Section 1.5: Solutions of Systems of Linear Equations

1. Graphs and Graphing Utilities

Since graphing calculators are not required for this course, we will use Desmos to graph equations.

Example. For a certain city, the cost C of obtaining drinking water with p percent impurities is modeled by

$$C(p) = \frac{120{,}000}{p} - 1200.$$

Because p represents a percentage, we must have

$$\boxed{0 < p \leq 100}.$$

Step 1: Identify the restriction.

The formula is undefined when $p = 0$, and a percentage cannot be negative or greater than 100. Therefore,

$$\boxed{0 < p \leq 100}.$$

Step 2: Enter the function in Desmos.

In Desmos, enter

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C(p)=120000/p-1200{0<p<=100}.
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The restriction tells Desmos to graph only the part of the curve where

$$0 < p \leq 100.$$

Step 3: Check a few values.

For example,

$$\begin{aligned} C(20) &= 4800, \\ C(40) &= 1800, \\ C(60) &= 800, \\ C(80) &= 300, \\ C(100) &= 0. \end{aligned}$$

Thus, as the percentage of impurities increases, the cost decreases.

$$\boxed{\text{The graph is decreasing on } 0 < p \leq 100.}$$

Key Idea: When graphing a function in an application, always check whether the variables have restrictions. Enter those restrictions in Desmos using braces { }.

2. Solutions of Systems of Linear Equations

Definition 2.1. A **system of equations** is a collection of two or more equations. An ordered pair (x, y) is a **solution** if it satisfies every equation in the system.

A system of linear equations has three possible outcomes:

$\boxed{\text{No solution (inconsistent)}}$ $\boxed{\text{Exactly one solution}}$ $\boxed{\text{Infinitely many solutions (dependent)}}$.

2.1 Solving Systems by Graphing

Example. Use graphing to find the solution(s) of each system.

1. No solution

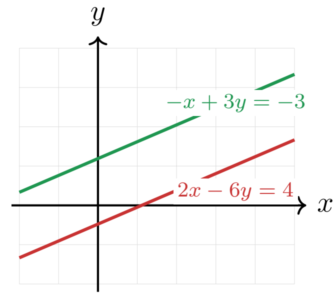
$$\begin{cases} 2x - 6y = 4, \\ -x + 3y = -3. \end{cases}$$

In slope-intercept form,

$$y = \frac{1}{3}x - \frac{2}{3}, \qquad y = \frac{1}{3}x + 1.$$

The lines have the same slope but different y-intercepts, so they are parallel.

$$\boxed{\text{No solution}}$$



2. One solution

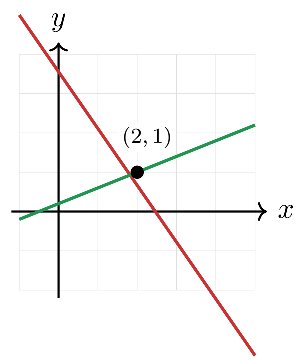
$$\begin{cases} 4x + 3y = 11, \\ 2x - 5y = -1. \end{cases}$$

In slope-intercept form,

$$y = -\frac{4}{3}x + \frac{11}{3}, \qquad y = \frac{2}{5}x + \frac{1}{5}.$$

The graphs intersect at exactly one point:

$$\boxed{(2, 1)}$$

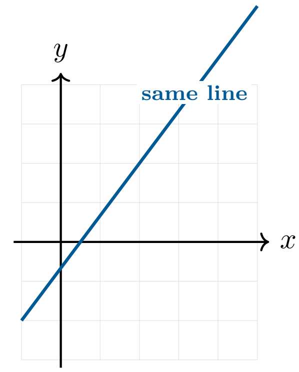


3. Infinitely many solutions

$$\begin{cases} -4x + 3y = -2, \\ 8x - 6y = 4. \end{cases}$$

The second equation is -2 times the first. Thus, both equations represent the same line.

$$\boxed{\text{Infinitely many solutions}}$$



One intersection → one solution

Parallel lines → no solution

Same line → infinitely many solutions

2.2 Equivalent Systems

Two systems are **equivalent** if they have the same solution set.

We can obtain an equivalent system by performing any of the following operations:

1. Replace an equation by an equivalent equation.
2. Interchange two equations.
3. Multiply an equation by a nonzero constant.
4. Add a multiple of one equation to another equation.

> **Key idea:** Equivalent systems may look different, but they have exactly the same solutions.

2.3 Substitution Method

The **substitution method** uses an equation to express one variable in terms of the other and then substitutes that expression into the remaining equation.

Example. Solve the system

$$\begin{cases} 2x + 3y = 4, \\ x - 2y = 3. \end{cases}$$

1. Solve one equation for one variable.
2. Substitute into the other equation.
3. Solve for the remaining variable.
4. Substitute back to find the other variable.
5. Check the solution.

Step 1: Solve for one variable.

From the second equation,

$$x = 2y + 3.$$

Step 2: Substitute.
Substitute $x = 2y + 3$ into the first equation:

$$2(2y + 3) + 3y = 4.$$

Step 3: Solve for y.

$$4y + 6 + 3y = 4 \implies 7y = -2 \implies y = -\frac{2}{7}.$$

Step 4: Substitute back to find x.

$$x = 2\left(-\frac{2}{7}\right) + 3 = \frac{17}{7}.$$

Therefore,

$$\boxed{(x, y) = \left(\frac{17}{7}, -\frac{2}{7}\right)}.$$

Step 5: Check.
Substitute $x = \frac{17}{7}$ and $y = -\frac{2}{7}$ into the original equations:

$$2\left(\frac{17}{7}\right) + 3\left(-\frac{2}{7}\right) = \frac{34 - 6}{7} = 4,$$

$$\frac{17}{7} - 2\left(-\frac{2}{7}\right) = \frac{17 + 4}{7} = 3.$$

Both equations are satisfied, so

$$\boxed{\left(\frac{17}{7}, -\frac{2}{7}\right)}$$

is the solution.