Lecture 5

Section 1.5: Continued and §1.6 Applications of Functions in Business and Economics

1. Elimination Method

The **elimination method** solves a system by combining the equations so that one of the variables is eliminated.

1. Multiply one or both equations by a nonzero constant so that the coefficients of one variable are opposites.
2. Add the equations to eliminate that variable.
3. Solve for the remaining variable.
4. Substitute back into one of the original equations to find the other variable.
5. Check the solution in both original equations.

Example. Solve the system

$$\begin{cases} 2x - 5y = 4, \\ x + 2y = 3. \end{cases}$$

Step 1: Make the coefficients of x opposites.
Multiply the second equation by -2:

$$\begin{cases} 2x - 5y = 4, \\ -2x - 4y = -6. \end{cases}$$

Step 2: Add the equations to eliminate x.

$$\begin{aligned} (2x - 5y) + (-2x - 4y) &= 4 + (-6), \\ 0x - 9y &= -2. \end{aligned}$$

Therefore,

$$y = \frac{2}{9}.$$

Step 3: Substitute back to find x.
Using $x + 2y = 3$,

$$x + 2\left(\frac{2}{9}\right) = 3,$$

so

$$x = \frac{23}{9}.$$

Therefore,

$$\boxed{(x, y) = \left(\frac{23}{9}, \frac{2}{9}\right)}.$$

Step 4: Check.

Substitute

$$x = \frac{23}{9}, \qquad y = \frac{2}{9}$$

into the original equations:

$$2\left(\frac{23}{9}\right) - 5\left(\frac{2}{9}\right) = \frac{46 - 10}{9} = 4,$$

$$\frac{23}{9} + 2\left(\frac{2}{9}\right) = \frac{27}{9} = 3.$$

Both equations are satisfied. Hence,

$$\boxed{\left(\frac{23}{9}, \frac{2}{9}\right)}$$

is the solution.

Practice questions on Elimination Method

Example. Use the elimination method to solve the following systems.

$$\boxed{\begin{cases} 3x - 6y = 9, \\ -2x + 4y = -6. \end{cases}}$$

$$\boxed{\begin{cases} 5x + 2y = 16, \\ 3x - 2y = 8. \end{cases}}$$

$$\boxed{\begin{cases} 4x - 6y = 12, \\ -8x + 12y = -20. \end{cases}}$$

Example. A woman has \$500,000 invested in two rental properties. One property yields an annual return of 10%, while the other yields an annual return of 12%. Her total annual return from the two investments is \$53,000.

Let

$$x = \text{amount invested at } 10\%, \qquad y = \text{amount invested at } 12\%.$$

Step 1: Write an equation for the total investment.

The sum of the two investments is \$500,000:

$$\boxed{x + y = 500{,}000}.$$

Step 2: Find the annual return from each investment.

The annual return is

$$\text{rate} \times \text{investment}.$$

Thus, the 10% investment earns

$$0.10x,$$

while the 12% investment earns

$$0.12y.$$

Step 3: Write an equation for the total annual return.
The total annual return is \$53,000:

$$\boxed{0.10x + 0.12y = 53{,}000}.$$

Therefore, we need to solve the system

$$\begin{cases} x + y = 500{,}000, \\ 0.10x + 0.12y = 53{,}000. \end{cases}$$

Step 4: Eliminate one variable.
Multiply the first equation by -0.10:

$$\begin{cases} -0.10x - 0.10y = -50{,}000, \\ 0.10x + 0.12y = 53{,}000. \end{cases}$$

Add the equations:

$$\begin{aligned} -0.10x - 0.10y + (0.10x + 0.12y) &= -50{,}000 + 53{,}000, \\ 0.02y &= 3{,}000. \end{aligned}$$

Thus,

$$y = 150{,}000.$$

Step 5: Find x.
Substitute $y = 150{,}000$ into $x + y = 500{,}000$:

$$x + 150{,}000 = 500{,}000,$$

so

$$x = 350{,}000.$$

Therefore,

$$\boxed{x = \$350{,}000, \qquad y = \$150{,}000}.$$

Step 6: Check.
Total investment:

$$350{,}000 + 150{,}000 = 500{,}000.$$

Total annual return:

$$0.10(350{,}000) + 0.12(150{,}000) = 35{,}000 + 18{,}000 = 53{,}000.$$

Both conditions are satisfied. Hence,

$$\boxed{\$350{,}000 \text{ is invested at } 10\%, \qquad \$150{,}000 \text{ is invested at } 12\%.}$$

Example. A nurse has two solutions containing different concentrations of a medication. One is a 12.5% concentration and the other is a 5% concentration. How many cubic centimeters of each should she mix to obtain 20 cubic centimeters of an 8% concentration?

Let

$$x = \text{cubic centimeters of the 12.5\% solution}, \qquad y = \text{cubic centimeters of the 5\% solution}.$$

Step 1: Write the system.
The total volume is 20 cc:

$$x + y = 20.$$

The total amount of medication is 8% of 20:

$$0.125x + 0.05y = 0.08(20).$$

Thus,

$$\begin{cases} x + y = 20, \\ 0.125x + 0.05y = 1.6. \end{cases}$$

Multiply the second equation by 1000:

$$\begin{cases} x + y = 20, \\ 125x + 50y = 1600. \end{cases}$$

Step 2: Eliminate y.
Multiply the first equation by -50:

$$\begin{cases} -50x - 50y = -1000, \\ 125x + 50y = 1600. \end{cases}$$

Add:

$$75x = 600.$$

Therefore,

$$x = 8.$$

Step 3: Find y.
Substitute $x = 8$ into $x + y = 20$:

$$8 + y = 20,$$

so

$$y = 12.$$

Therefore,

$$\boxed{x = 8 \text{ cc}} \quad \text{and} \quad \boxed{y = 12 \text{ cc}}.$$

Answer: Mix

$$\boxed{\text{8 cc of the 12.5\% solution}}$$

with

$$\boxed{\text{12 cc of the 5\% solution}}.$$

1.1 Applications of Systems of Equations

Systems of equations can be used to model problems involving **investments, mixtures, resources, and other quantities**. The key is to identify the unknowns and translate the information given in the problem into equations.

Application: Comparing Linear Models

Example. The annual sales of two companies are modeled by

$$S_1(x) = 32.5x + 420, \qquad S_2(x) = 18.5x + 700,$$

where x represents the number of years after 2020 and the sales are measured in millions of dollars. In what year will the two companies have the same annual sales? What will those sales be?

Step 1: Set the two models equal.

At the point where the sales are equal,

$$32.5x + 420 = 18.5x + 700.$$

Step 2: Solve for x.

$$32.5x - 18.5x = 700 - 420,$$
$$14x = 280,$$

so

$$x = 20.$$

Thus, the models predict equal sales 20 years after 2020:

$$2020 + 20 = \boxed{2040}.$$

Step 3: Find the common sales amount.

Substitute $x = 20$ into either model:

$$S_1(20) = 32.5(20) + 420 = 1070.$$

Therefore,

$$\boxed{\$1.07 \text{ billion}}$$

is the predicted annual sales for each company.

Answer:

$$\boxed{2040 \quad \text{and} \quad \$1.07 \text{ billion}}$$

Application: Investments

Example. A student invests \$40,000 in two accounts. One account earns 6% annual interest and the other earns 9%. If the total annual interest is \$3,000, how much is invested in each account?

Let

$$x = \text{amount invested at } 6\%, \qquad y = \text{amount invested at } 9\%.$$

Step 1: Write the system.

The total investment is \$40,000:

$$x + y = 40{,}000.$$

The total annual interest is \$3,000:

$$0.06x + 0.09y = 3{,}000.$$

Thus,

$$\boxed{\begin{cases} x + y = 40{,}000, \\ 0.06x + 0.09y = 3{,}000. \end{cases}}$$

Step 2: Eliminate x.
Multiply the first equation by -0.06:

$$\begin{cases} -0.06x - 0.06y = -2{,}400, \\ 0.06x + 0.09y = 3{,}000. \end{cases}$$

Add:

$$-0.06x - 0.06y + (0.06x + 0.09y) = -2{,}400 + 3{,}000,$$
$$0.03y = 600.$$

Therefore,

$$y = 20{,}000.$$

Step 3: Find x.

$$x + 20{,}000 = 40{,}000,$$

so

$$x = 20{,}000.$$

Therefore,

$$\boxed{x = \$20{,}000, \qquad y = \$20{,}000}.$$

Check:

$$0.06(20{,}000) + 0.09(20{,}000) = 1{,}200 + 1{,}800 = 3{,}000.$$

Answer:

$$\boxed{\$20{,}000 \text{ at } 6\%} \qquad \boxed{\$20{,}000 \text{ at } 9\%}.$$

Application: Resource Allocation

Example. A farm produces two types of animal feed. Each bag of Feed A requires 4 units of corn and 3 units of soybean meal. Each bag of Feed B requires 2 units of corn and 5 units of soybean meal. The farm has exactly 32 units of corn and 34 units of soybean meal available. How many bags of each feed should be produced if all of the resources are used?
Let

$$x = \text{number of bags of Feed A}, \qquad y = \text{number of bags of Feed B}.$$

Step 1: Write the system.
Corn:

$$4x + 2y = 32.$$

Soybean meal:

$$3x + 5y = 34.$$

Thus,

$$\boxed{\begin{cases} 4x + 2y = 32, \\ 3x + 5y = 34. \end{cases}}$$

Step 2: Eliminate y.
Multiply the equations by appropriate constants:

$$\begin{cases} 20x + 10y = 160, \\ -6x - 10y = -68. \end{cases}$$

Add:

$$14x = 92.$$

Therefore,

$$x = \frac{46}{7}.$$

Since the problem asks for a number of bags, this is not a whole number. Thus, the given resource amounts cannot be completely used with whole bags.

$$\boxed{\text{No whole-number solution}}$$

Remark. In applications, a mathematical solution must also make sense in the context of the problem. Quantities such as people, machines, or whole packages may need to be integers.

Application: Mixtures

Example. A laboratory has two solutions of acid. One solution is 30% acid and the other is 10% acid. How many liters of each solution should be mixed to obtain 20 liters of a 25% solution?
Let

$$x = \text{liters of the 30\% solution}, \qquad y = \text{liters of the 10\% solution}.$$

Step 1: Write the system.
The total volume is 20 liters:

$$x + y = 20.$$

The amount of pure acid in the final mixture is 25% of 20, or 5 liters:

$$0.30x + 0.10y = 5.$$

Thus,

$$\boxed{\begin{cases} x + y = 20, \\ 0.30x + 0.10y = 5. \end{cases}}$$

Step 2: Eliminate y.
Multiply the second equation by 10:

$$\begin{cases} x + y = 20, \\ 3x + y = 50. \end{cases}$$

Multiply the first equation by -1:

$$\begin{cases} -x - y = -20, \\ 3x + y = 50. \end{cases}$$

Add:

$$2x = 30.$$

Therefore,

$$x = 15.$$

Step 3: Find y.

$$15 + y = 20,$$

so

$$y = 5.$$

Therefore,

$$\boxed{15 \text{ L of the } 30\% \text{ solution}}$$

and

$$\boxed{5 \text{ L of the } 10\% \text{ solution}}$$

are needed.

Check:

$$0.30(15) + 0.10(5) = 4.5 + 0.5 = 5,$$

and

$$\frac{5}{20} = 0.25 = 25\%.$$

Thus, the mixture has the required concentration.

Answer:

$$\boxed{15 \text{ L of } 30\% \quad \text{and} \quad 5 \text{ L of } 10\%}.$$

Application: Three-Variable Systems

Example. A meal contains three food supplements A, B, and C. Each ounce of A provides 4 units of a nutrient, each ounce of B provides 7 units, and each ounce of C provides 10 units. The meal must contain exactly 50 ounces of food and provide exactly 300 units of the nutrient.

Suppose the amount of C is twice the amount of A. How many ounces of each supplement should be included?

Let

$$x = \text{ounces of } A, \qquad y = \text{ounces of } B, \qquad z = \text{ounces of } C.$$

The three conditions give

$$\boxed{\begin{cases} x + y + z = 50, \\ 4x + 7y + 10z = 300, \\ z = 2x. \end{cases}}$$

Using $z = 2x$, substitute into the first two equations:

$$x + y + 2x = 50,$$

$$4x + 7y + 20x = 300.$$

Thus,

$$\begin{cases} 3x + y = 50, \\ 24x + 7y = 300. \end{cases}$$

From the first equation,

$$y = 50 - 3x.$$

Substitute:

$$24x + 7(50 - 3x) = 300,$$

$$24x + 350 - 21x = 300,$$

$$3x = -50.$$

Therefore,

$$x = -\frac{50}{3}.$$

Because an amount of food cannot be negative, this system has

$$\boxed{\text{no physically meaningful solution}}.$$

Remark. A solution to a mathematical system must also satisfy the conditions of the real-world problem. Negative amounts of food, money, or medication are not meaningful.