Lecture 6

Section 2.1: Quadratic Equations

1. Quadratic Equations

Definition 1.1. A **quadratic equation in one variable** is an equation of second degree that can be written in the general form

$$\boxed{ax^2 + bx + c = 0}, \qquad a \neq 0,$$

where a, b, and c are constants.

The coefficient a cannot be zero. Otherwise, the equation would no longer be quadratic.

Examples of quadratic equations:

$$x^2 - 5x + 6 = 0,$$

$$2x^2 + 7x - 4 = 0,$$

$$-3x^2 + 8 = 0.$$

Not quadratic:

$$3x + 5 = 0,$$

because the highest power of x is 1.

1.1 Zero Product Property

Factoring is one of the most useful methods for solving quadratic equations.
The key idea is the **Zero Product Property**.

Definition 1.2. For real numbers a and b,

$$ab = 0$$

if and only if

$$a = 0 \qquad \text{or} \qquad b = 0$$

(or both).

In other words,

$$\boxed{ab = 0 \quad \Longrightarrow \quad a = 0 \text{ or } b = 0.}$$

This property allows us to turn a factored quadratic equation into two simpler equations.

Example: Solving by the Zero Product Property

$$x(x + 3) = 0.$$

Since the product is zero,

$$x = 0 \qquad \text{or} \qquad x + 3 = 0.$$

Therefore,

$$\boxed{x = 0,\ -3}.$$

Example. Solve

$$(x-4)(3x+1) = 0.$$

Set each factor equal to zero:

$$x - 4 = 0 \qquad \text{or} \qquad 3x + 1 = 0.$$

Thus,

$$x = 4 \qquad \text{or} \qquad x = -\frac{1}{3}.$$

Therefore,

$$\boxed{x = 4,\ -\frac{1}{3}}.$$

2. Solving Quadratic Equations by Factoring

To solve a quadratic equation by factoring:

1. Rewrite the equation in the general form
$$ax^2 + bx + c = 0.$$
2. Factor the quadratic expression.
3. Use the Zero Product Property.
4. Solve each resulting linear equation.
5. Check the solutions when appropriate.

Example. Solve

$$2x^2 + x = 3x + 12.$$

Step 1: Rewrite in general form.
Move everything to one side:

$$2x^2 + x - 3x - 12 = 0.$$

Therefore,

$$2x^2 - 2x - 12 = 0.$$

Factor out 2:

$$2(x^2 - x - 6) = 0.$$

Step 2: Factor.
We need two numbers whose product is -6 and whose sum is -1:

$$-3 \cdot 2 = -6, \qquad -3 + 2 = -1.$$

Thus,

$$2(x-3)(x+2) = 0.$$

Step 3: Use the Zero Product Property.
Since $2 \neq 0$,

$$x - 3 = 0 \qquad \text{or} \qquad x + 2 = 0.$$

Therefore,

$$x = 3 \qquad \text{or} \qquad x = -2.$$

Hence,

$$\boxed{x = 3,\, -2}.$$

Factoring by Grouping
Sometimes the middle term must first be split into two terms.
Example. Solve

$$2x^2 - 2x - 12 = 0.$$

The product of a and c is

$$ac = (2)(-12) = -24.$$

We need two numbers whose product is -24 and whose sum is -2:

$$-6 \cdot 4 = -24, \qquad -6 + 4 = -2.$$

Rewrite the middle term:

$$2x^2 - 6x + 4x - 12 = 0.$$

Group:

$$(2x^2 - 6x) + (4x - 12) = 0.$$

Factor each group:

$$2x(x-3) + 4(x-3) = 0.$$

Factor out the common factor:

$$(x-3)(2x+4) = 0.$$

Apply the Zero Product Property:

$$x - 3 = 0 \qquad \text{or} \qquad 2x + 4 = 0.$$

Thus,

$$\boxed{x = 3,\ -2}.$$

2.1 Practice: Solve by Factoring

Example. Solve the following quadratic equations by factoring.

1.

$$(x+3)(x-1) = 5$$

2.

$$-4x^2 + 8x - 3 = 0$$

Helpful reminder:
Before factoring, make sure the equation is written as

$$\boxed{ax^2 + bx + c = 0}.$$

3. Solving Equations of the Form $x^2 = C$

Another useful form of a quadratic equation is

$$x^2 = C.$$

Taking the square root of both sides gives

$$\boxed{x = \pm\sqrt{C}}.$$

The symbol $\pm$ is important because both the positive and negative square roots must be considered.
Example. Solve

$$(x-1)^2 = 9.$$

Take the square root of both sides:

$$x - 1 = \pm\sqrt{9}.$$

Therefore,

$$x - 1 = \pm 3.$$

This gives two equations:

$$x - 1 = 3 \qquad \text{or} \qquad x - 1 = -3.$$

Hence,

$$x = 4 \qquad \text{or} \qquad x = -2.$$

Therefore,

$$\boxed{x = 4,\ -2}.$$

Example. Solve

$$4x^2 - 1 = 0.$$

Move 1 to the other side:

$$4x^2 = 1.$$

Divide by 4:

$$x^2 = \frac{1}{4}.$$

Take the square root:

$$x = \pm\sqrt{\frac{1}{4}}.$$

Thus,

$$\boxed{x = \pm\frac{1}{2}}.$$

4. The Quadratic Formula

Not every quadratic equation factors easily.

The **quadratic formula** provides a method for solving any quadratic equation of the form

$$ax^2 + bx + c = 0, \qquad a \neq 0.$$

Definition 4.1. For

$$ax^2 + bx + c = 0,$$

the solutions are given by

$$\boxed{x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}}.$$

The quadratic formula can be used when factoring is difficult or when a quadratic does not factor nicely over the integers.

4.1 Using the Quadratic Formula

Example. Solve

$$2x^2 - 3x - 2 = 0.$$

Identify the coefficients:

$$a = 2, \qquad b = -3, \qquad c = -2.$$

Substitute into the quadratic formula:

$$x = \frac{-(-3) \pm \sqrt{(-3)^2 - 4(2)(-2)}}{2(2)}.$$

Simplify:

$$x = \frac{3 \pm \sqrt{9+16}}{4}.$$

Thus,

$$x = \frac{3 \pm 5}{4}.$$

This gives

$$x = \frac{3+5}{4} = 2$$

or

$$x = \frac{3-5}{4} = -\frac{1}{2}.$$

Therefore,

$$\boxed{x = 2,\ -\frac{1}{2}}.$$

5. The Discriminant

The expression under the square root in the quadratic formula is

$$b^2 - 4ac.$$

This quantity is called the **discriminant**.

Definition 5.1. For a quadratic equation

$$ax^2 + bx + c = 0,$$

the **discriminant** is

$$\boxed{\Delta = b^2 - 4ac}.$$

The discriminant tells us how many real solutions the quadratic equation has.

- If
  $$b^2 - 4ac > 0,$$
  the equation has exactly **two distinct real solutions**.
- If
  $$b^2 - 4ac = 0,$$
  the equation has exactly **one real solution**.
- If
  $$b^2 - 4ac < 0,$$
  the equation has **no real solutions**.

Example: Two Real Solutions:Determine the number of real solutions of

$$x^2 - 5x + 6 = 0.$$

Here,

$$a = 1, \qquad b = -5, \qquad c = 6.$$

The discriminant is

$$\Delta = (-5)^2 - 4(1)(6) = 25 - 24 = 1.$$

Since

$$\Delta > 0,$$

the equation has

$$\boxed{\text{two distinct real solutions}}.$$

Example: One Real Solution: Determine the number of real solutions of

$$x^2 - 6x + 9 = 0.$$

Here,

$$a = 1, \qquad b = -6, \qquad c = 9.$$

The discriminant is

$$\Delta = (-6)^2 - 4(1)(9) = 36 - 36 = 0.$$

Therefore,

$$\boxed{\text{one real solution}}.$$

Example: No Real Solutions: Determine the number of real solutions of

$$x^2 + 4x + 8 = 0.$$

Here,

$$a = 1, \qquad b = 4, \qquad c = 8.$$

The discriminant is

$$\Delta = 4^2 - 4(1)(8) = 16 - 32 = -16.$$

Since

$$\Delta < 0,$$

there are

$$\boxed{\text{no real solutions}}.$$

6. Application: Height of a Ball

Quadratic functions often arise when modeling the height of an object moving vertically.

Example. Suppose some hooligans kick a ball up in the air from the roof of the library. Assume the height, in feet, of the ball t seconds after being kicked is

$$\boxed{h(t) = -32t^2 + 64t + 40}.$$

We can use this quadratic model to determine when the ball reaches different heights.

6.1 When Is the Ball 80 Feet High?

We want

$$h(t) = 80.$$

Substitute the height into the model:

$$-32t^2 + 64t + 40 = 80.$$

Move everything to one side:

$$-32t^2 + 64t - 40 = 0.$$

Divide by -8:

$$4t^2 - 8t + 5 = 0.$$

Use the quadratic formula:

$$t = \frac{-(-8) \pm \sqrt{(-8)^2 - 4(4)(5)}}{2(4)}.$$

The discriminant is

$$64 - 80 = -16.$$

Since the discriminant is negative, there are no real solutions.
Therefore,

$$\boxed{\text{The ball is never 80 feet high.}}$$

In fact, the maximum height of the ball is less than 80 feet.

6.2 When Is the Ball 72 Feet High?

We want

$$h(t) = 72.$$

Thus,

$$-32t^2 + 64t + 40 = 72.$$

Move everything to one side:

$$-32t^2 + 64t - 32 = 0.$$

Divide by -32:

$$t^2 - 2t + 1 = 0.$$

Factor:

$$(t-1)^2 = 0.$$

Therefore,

$$\boxed{t = 1}.$$

So the ball is 72 feet high exactly

$$\boxed{\text{1 second}}$$

after it is kicked.

6.3 When Is the Ball 40 Feet High?

We want

$$h(t) = 40.$$

Thus,

$$-32t^2 + 64t + 40 = 40.$$

Simplify:

$$-32t^2 + 64t = 0.$$

Factor:

$$-32t(t-2) = 0.$$

By the Zero Product Property,

$$t = 0 \qquad \text{or} \qquad t = 2.$$

Therefore,

$$\boxed{t = 0,\ 2}.$$

The ball is 40 feet high at the instant it is kicked and again 2 seconds later.

6.4 When Does the Ball Hit the Ground?

The ball hits the ground when its height is zero:

$$h(t) = 0.$$

Thus,

$$-32t^2 + 64t + 40 = 0.$$

Divide by -8:

$$4t^2 - 8t - 5 = 0.$$

Use the quadratic formula:

$$t = \frac{-(-8) \pm \sqrt{(-8)^2 - 4(4)(-5)}}{2(4)}.$$

Simplify:

$$t = \frac{8 \pm \sqrt{64 + 80}}{8}.$$

Therefore,

$$t = \frac{8 \pm \sqrt{144}}{8} = \frac{8 \pm 12}{8}.$$

This gives

$$t = \frac{20}{8} = \frac{5}{2}$$

or

$$t = \frac{-4}{8} = -\frac{1}{2}.$$

Mathematically, there are two roots:

$$t = \frac{5}{2} \qquad \text{and} \qquad t = -\frac{1}{2}.$$

However, negative time does not make sense in this context because $t = 0$ represents the moment the ball is kicked.

Therefore, the physically meaningful answer is

$$\boxed{t = \frac{5}{2} = 2.5 \text{ seconds}}.$$

So the ball hits the ground approximately

$$\boxed{\text{2.5 seconds after being kicked.}}$$