Lecture 7

Section 2.2: Quadratic Functions: Parabolas

1. Quadratic Functions and the Parabola

A **quadratic function** is a degree-two polynomial function of the standard form:

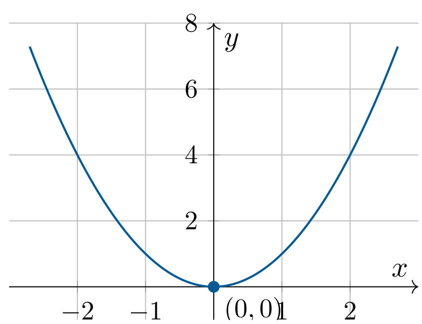
$$\boxed{f(x) = ax^2 + bx + c, \qquad a \neq 0,}$$

where a, b, and c are real numbers. Examples include:

$$f(x) = x^2, \qquad f(x) = 2x^2 + 3x - 1, \qquad f(x) = -4x^2 + 8x + 5.$$

The graph of any quadratic function is a smooth, continuous, U-shaped curve called a **parabola**. The simplest quadratic function is the parent function:

$$\boxed{f(x) = x^2}.$$



The graph is symmetric about the y-axis (the line $x = 0$).

2. Properties of the Coefficient a

2.1 Opening Direction

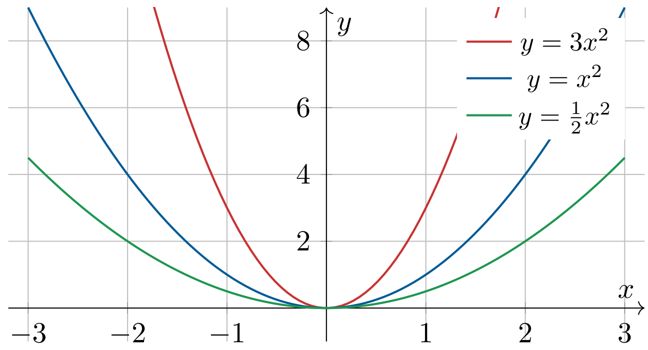
The sign of the leading coefficient a determines whether the parabola opens upward or downward:

$$\boxed{a > 0 \Rightarrow \text{opens upward } (\cup)} \qquad \boxed{a < 0 \Rightarrow \text{opens downward } (\cap)}$$

2.2 Width of a Parabola

The absolute value $|a|$ governs vertical dilation (stretch or compression):

$$\boxed{\text{Larger } |a| \Rightarrow \text{narrower (steeper)}} \qquad \boxed{\text{Smaller } |a| \Rightarrow \text{wider (flatter)}}$$



3. Vertex and Axis of Symmetry

Every parabola has a turning point where it changes direction, known as the **vertex**.

The parabola is completely symmetric across a vertical line passing through the vertex called the **axis of symmetry**:

$$\boxed{x = -\frac{b}{2a}}.$$

> **Definition.**
> The quadratic function $y = f(x) = ax^2 + bx + c$ has its **vertex** at
>
> $$\left(-\frac{b}{2a},\ f\left(-\frac{b}{2a}\right)\right).$$
>
> The optimal value occurs at the vertex of a parabola:
>
> - A **maximum** if $a < 0$ $\frown$
> - A **minimum** if $a > 0$ $\smile$

4. Intercepts of a Parabola

4.1 y-Intercept

The y-intercept occurs where the graph crosses the vertical axis ($x = 0$):

$$f(0) = a(0)^2 + b(0) + c = c \implies \boxed{y\text{-intercept } = (0, c)}.$$

Quick Example: For $f(x) = 2x^2 - 5x + 7$, evaluating $f(0)$ gives 7. Thus, the y-intercept is $(\mathbf{0}, \mathbf{7})$.

4.2 x-Intercepts (Zeros)

The x-intercepts occur where the graph meets the horizontal axis ($y = 0$). To find them, solve the quadratic equation:

$$\boxed{f(x) = 0 \iff ax^2 + bx + c = 0}.$$

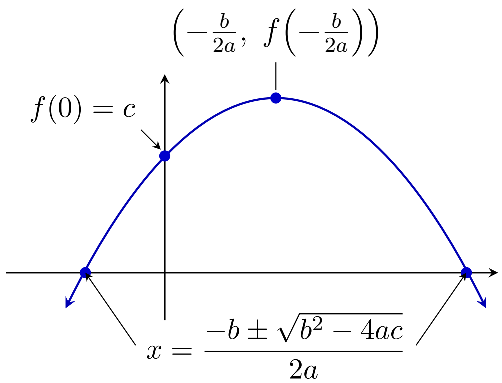
These solutions can be found by factoring or by using the quadratic formula:

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}.$$

Quick Example: For $f(x) = x^2 - 5x + 6$, setting $f(x) = 0$ gives:

$$x^2 - 5x + 6 = 0 \implies (x-2)(x-3) = 0 \implies x = 2 \quad \text{or} \quad x = 3.$$

Thus, the x-intercepts are $(\mathbf{2}, \mathbf{0})$ and $(\mathbf{3}, \mathbf{0})$.



5. Comprehensive Worked Example

Example 1. Consider the function:

$$y = 6x - x^2.$$

(a) Rewrite the function in standard form $y = ax^2 + bx + c$, and identify a, b, and c.

(b) Does the parabola open upward or downward? Is the vertex a maximum or minimum?

(c) Find the coordinates (x, y) of the vertex.

(d) Find all zeros (x-intercepts).

(e) Find the y-intercept.

(f) Sketch the graph of the function.

Solution.

Part (a): Standard Form and Coefficients
Rearrange terms in descending powers of x:

$$y = -x^2 + 6x + 0 \implies \mathbf{a = -1}, \quad \mathbf{b = 6}, \quad \mathbf{c = 0}.$$

Part (b): Opening Direction and Extremum
Since $a = -1 < 0$ (negative), the parabola **opens downward** ($\cap$).
Therefore, the vertex represents an absolute **maximum point**.

Part (c): Coordinates of the Vertex
Calculate the x-coordinate using the vertex formula:

$$x = -\frac{b}{2a} = -\frac{6}{2(-1)} = -\frac{6}{-2} = \mathbf{3}.$$

Substitute $x = 3$ into the original equation to obtain y:

$$y = 6(3) - (3)^2 = 18 - 9 = \mathbf{9}.$$

Thus, the vertex is $(\mathbf{3}, \mathbf{9})$, and the axis of symmetry is the line $\mathbf{x = 3}$.

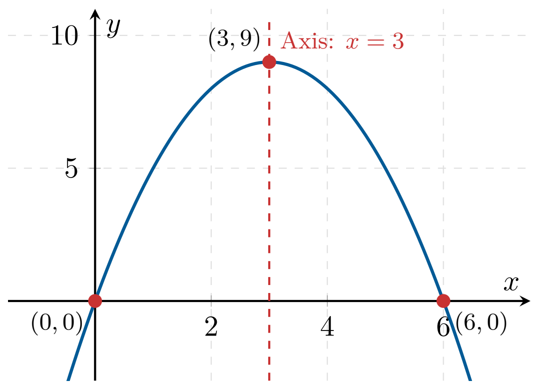
Part (d): Zeros (x-Intercepts)
Set $y = 0$ and factor:

$$6x - x^2 = 0 \implies x(6 - x) = 0 \implies x = 0 \quad \text{or} \quad x = 6.$$

The zeros are $\mathbf{x = 0}$ and $\mathbf{x = 6}$, giving the points $(\mathbf{0}, \mathbf{0})$ and $(\mathbf{6}, \mathbf{0})$.
Part (e): y-Intercept
Substitute $x = 0$:

$$y = 6(0) - (0)^2 = \mathbf{0} \implies (\mathbf{0}, \mathbf{0}).$$

Part (f): Graph Sketch



Example 2

The monthly profit from the sale of a product is given by

$$P(x) = 36x - 0.3x^2 - 240$$

dollars, where x is the number of units produced and sold each month.

(a) What level of production maximizes the monthly profit?

(b) What is the maximum possible monthly profit?

Solution
The monthly profit is

$$P(x) = 36x - 0.3x^2 - 240.$$

Rewrite the function in standard quadratic form:

$$P(x) = -0.3x^2 + 36x - 240.$$

This is a quadratic function of the form

$$P(x) = ax^2 + bx + c,$$

where

$$a = -0.3, \qquad b = 36, \qquad c = -240.$$

Since

$$a = -0.3 < 0,$$

the parabola opens downward. Therefore, the vertex represents the **maximum profit**.

(a) Find the production level that maximizes profit.
For a quadratic function

$$P(x) = ax^2 + bx + c,$$

the x-coordinate of the vertex is

$$x = -\frac{b}{2a}.$$

Substitute $a = -0.3$ and $b = 36$:

$$x = -\frac{36}{2(-0.3)}.$$

Simplify:

$$x = -\frac{36}{-0.6}$$

$$x = 60.$$

Therefore, the profit is maximized when

$$\boxed{60 \text{ units}}$$

are produced and sold each month.

(b) Find the maximum possible profit.
To find the maximum profit, substitute $x = 60$ into the profit function:

$$P(60) = 36(60) - 0.3(60)^2 - 240.$$

Calculate each term:

$$36(60) = 2160$$

and

$$60^2 = 3600.$$

Thus,

$$P(60) = 2160 - 0.3(3600) - 240.$$

Since

$$0.3(3600) = 1080,$$

we get

$$P(60) = 2160 - 1080 - 240.$$

Therefore,

$$P(60) = 840.$$

Hence, the maximum possible monthly profit is

$$\boxed{\$840}.$$