Lecture 7

What's Unusual? The Empirical Rule and z-Scores

§3.2 (continued)

1. z-Scores

The Empirical Rule tells us whether an observation is close to or far from the mean in terms of standard deviations.

A z-**score** gives us a precise numerical measure of this distance.

Definition.
A z-**score** measures how many standard deviations an observed value x is from the mean.
The formula is

$$\boxed{z = \frac{x - \bar{x}}{s}}$$

where

- x is the observed data value;
- $\bar{x}$ is the sample mean;
- s is the sample standard deviation.

1.1 Interpreting a z-Score

The sign of the z-score tells us whether the observation is above or below the mean.

- If $z > 0$, the observation is above the mean.
- If $z < 0$, the observation is below the mean.
- If $z = 0$, the observation is exactly equal to the mean.

The absolute value tells us how far the observation is from the mean.
For example,

$$z = 2$$

means that the observation is two standard deviations above the mean.
Similarly,

$$z = -2$$

means that the observation is two standard deviations below the mean.

$$\boxed{z = 2 \quad \Longrightarrow \quad \text{2 standard deviations above the mean}}$$

$$\boxed{z = -2 \quad \Longrightarrow \quad \text{2 standard deviations below the mean}}$$

Example: Heights of Men

Suppose a data set contains the heights, in inches, of 247 men.
The mean height is

$$\bar{x} = 70$$

inches, and the standard deviation is

$$s = 3$$

inches.
We can use z-scores to determine how unusual different heights are.

A Man Taller Than Two Standard Deviations Above the Mean

We want to determine which men have

$$z > 2.$$

Using the z-score formula,

$$z = \frac{x - 70}{3}.$$

Thus,

$$\frac{x - 70}{3} > 2.$$

Multiplying by 3,

$$x - 70 > 6.$$

Therefore,

$$x > 76.$$

So men taller than 76 inches have z-scores greater than 2.

$$\boxed{z > 2 \quad \Longleftrightarrow \quad x > 76 \text{ inches}}$$

The exact number of men is determined by counting the observations above 76 inches on the dotplot.

A Man More Than Two Standard Deviations Below the Mean

We want to determine which men have

$$z < -2.$$

Using the formula,

$$\frac{x - 70}{3} < -2.$$

Multiplying by 3,

$$x - 70 < -6.$$

Therefore,

$$x < 64.$$

Thus,

$$\boxed{z < -2 \quad \Longleftrightarrow \quad x < 64 \text{ inches}}$$

The exact number of men is determined by counting the observations below 64 inches on the dotplot.

A Man Who Is 75 Inches Tall

For a man who is 75 inches tall,

$$x = 75.$$

Therefore,

$$z = \frac{75 - 70}{3}.$$

Thus,

$$z = \frac{5}{3} \approx \boxed{1.67}.$$

A man who is 75 inches tall has a z-score of approximately

$$\boxed{z = 1.67}.$$

Interpretation: his height is approximately 1.67 standard deviations above the mean height.

2. Using z-Scores to Compare Observations

A major advantage of z-scores is that they allow us to compare observations measured on different scales.

The original scores may have different means and standard deviations, but z-scores put the observations on the same scale.

When comparing two observations, the observation with the **larger z-score** performed better relative to its group.

$$\boxed{\text{Larger } z \quad \Longrightarrow \quad \text{better relative performance}}$$

Example: Maria's Statistics Exams

On which exam did Maria perform better when compared to the whole class?

The class statistics for the two exams were:

	Maria's Score	Mean	Standard Deviation
Exam 1	80	70	10
Exam 2	85	80	5

Although Maria received a higher raw score on Exam 2, we need to account for the different means and standard deviations.

Step 1: Calculate Maria's z-Score on Exam 1

For Exam 1,

$$x = 80, \qquad \bar{x} = 70, \qquad s = 10.$$

Therefore,

$$z_1 = \frac{80 - 70}{10}.$$

Thus,

$$z_1 = \frac{10}{10} = \boxed{1}.$$

Maria scored one standard deviation above the class mean on Exam 1.

Step 2: Calculate Maria's z-Score on Exam 2

For Exam 2,

$$x = 85, \qquad \bar{x} = 80, \qquad s = 5.$$

Therefore,

$$z_2 = \frac{85 - 80}{5}.$$

Thus,

$$z_2 = \frac{5}{5} = \boxed{1}.$$

Maria scored one standard deviation above the class mean on Exam 2.

Step 3: Compare the z-Scores

We found

$$z_1 = 1 \qquad \text{and} \qquad z_2 = 1.$$

Therefore, Maria performed equally well relative to her classmates on the two exams.

Answer:
Maria performed **equally well relative to the class** on the two exams.

$$\boxed{z_1 = z_2 = 1}$$

On both exams, Maria scored exactly one standard deviation above the class mean.
Although her raw score was higher on Exam 2, her relative performance was the same on both exams.