# Lecture 8

## Practice Problems on the Empirical Rule and $z$-Scores

## 1. Practice Problem 1: Gestation Periods

Distributions of gestation periods (lengths of pregnancy) for a particular animal are roughly bell-shaped. The mean gestation period for this animal is

$$\bar{x} = 288 \text{ days}$$

and the standard deviation is

$$s = 5 \text{ days}.$$

Which is more unusual, a baby being born 5 days early or a baby being born 5 days late?

### Step 1: Find the Two Gestation Periods

A baby born 5 days early has a gestation period of

$$288 - 5 = 283 \text{ days}.$$

A baby born 5 days late has a gestation period of

$$288 + 5 = 293 \text{ days}.$$

### Step 2: Calculate the $z$-Scores

For 283 days,

$$z = \frac{283 - 288}{5} = \frac{-5}{5} = \boxed{-1.00}.$$

For 293 days,

$$z = \frac{293 - 288}{5} = \frac{5}{5} = \boxed{1.00}.$$

### Multiple-Choice Answers

**A.** A baby being born 5 days late is more unusual because its corresponding $z$-score, $\boxed{1.00}$, is further from 0 than the corresponding $z$-score of $\boxed{-1.00}$ for a baby being born 5 days early.

**B.** A baby being born 5 days early is more unusual because its corresponding $z$-score, $\boxed{-1.00}$, is further from 0 than the corresponding $z$-score of $\boxed{1.00}$ for a baby being born 5 days late.

**C.** Both events are equally likely.

Since

$$|-1.00| = |1.00| = 1,$$

the two events are equally unusual.

**Correct Answer: C.**
Both events are equally likely.
The baby born 5 days early has $z = -1.00$, while the baby born 5 days late has $z = 1.00$. Both are exactly one standard deviation from the mean.

## 2. Practice Problem 2: Pollution Indices in Eastern Cities

In 2017, a pollution index was calculated for a sample of cities in the eastern states using data on air and water pollution.
Assume the distribution of pollution indices is unimodal and symmetric.
The mean of the distribution was

$$\bar{x} = 47.3 \text{ points}$$

with a standard deviation of

$$s = 11.6 \text{ points.}$$

### Part (a): Pollution Index Between 24.1 and 70.5

What percentage of eastern cities would you expect to have a pollution index between 24.1 and 70.5 points?
For the lower endpoint,

$$47.3 - 24.1 = 23.2.$$

Since

$$2(11.6) = 23.2,$$

the value 24.1 is exactly 2 standard deviations below the mean.
For the upper endpoint,

$$70.5 - 47.3 = 23.2.$$

Thus, 70.5 is exactly 2 standard deviations above the mean.
Therefore,

$$24.1 = \bar{x} - 2s$$

and

$$70.5 = \bar{x} + 2s.$$

By the Empirical Rule, approximately 95% of observations lie within 2 standard deviations of the mean.

**Answer:**

$$\boxed{95\%}$$

### Part (b): Pollution Index Between 35.7 and 58.9

What percentage of eastern cities would you expect to have a pollution index between 35.7 and 58.9 points?

For the lower endpoint,

$$47.3 - 35.7 = 11.6.$$

For the upper endpoint,

$$58.9 - 47.3 = 11.6.$$

Therefore,

$$35.7 = \bar{x} - s$$

and

$$58.9 = \bar{x} + s.$$

The interval is exactly one standard deviation below to one standard deviation above the mean.

By the Empirical Rule, approximately 68% of observations lie within 1 standard deviation of the mean.

**Answer:**

$$\boxed{68\%}$$

### Part (c): Pollution Index of 55.1

The pollution index for an eastern city in 2017 was 55.1 points. Based on this distribution, was this unusually high? Explain.

First, calculate the $z$-score:

$$z = \frac{x - \bar{x}}{s}.$$

Substituting,

$$z = \frac{55.1 - 47.3}{11.6}.$$

Thus,

$$z = \frac{7.8}{11.6} \approx \boxed{0.67}.$$

The value is only 0.67 standard deviations above the mean, so it is not unusually high.

### Multiple-Choice Answers

**A.** Yes, because 55.1 falls within three standard deviations of the mean, so it is therefore an unusually high pollution index.

**B.** No, because 55.1 falls exactly two standard deviations away from the mean, and it is therefore not an unusually high pollution index.

**C.** No, because 55.1 falls within two standard deviations away from the mean, and it is therefore not an unusually high pollution index.

**D.** Yes, because 55.1 falls within two standard deviations of the mean, so it is therefore an unusually high pollution index.

**Correct Answer: C.**
No, because 55.1 falls within two standard deviations away from the mean, and it is therefore not an unusually high pollution index.
Indeed,

$$z \approx 0.67,$$

which is well within two standard deviations of the mean.

## 3. Practice Problem 3: Comparing IQ Scores

An IQ test has a mean of

$$\bar{x} = 103$$

and a standard deviation of

$$s = 15.$$

Which is more unusual, an IQ of 133 or an IQ of 71?

### Step 1: Calculate the $z$-Score for an IQ of 133

For

$$x = 133,$$

we have

$$z = \frac{133 - 103}{15}.$$

Therefore,

$$z = \frac{30}{15} = \boxed{2.00}.$$

### Step 2: Calculate the $z$-Score for an IQ of 71

For

$$x = 71,$$

we have

$$z = \frac{71 - 103}{15}.$$

Therefore,

$$z = \frac{-32}{15} \approx \boxed{-2.13}.$$

**Multiple-Choice Answers**

**A.** An IQ of 71 is more unusual because its corresponding $z$-score, $\boxed{-2.13}$, is further from 0 than the corresponding $z$-score of $\boxed{2.00}$ for an IQ of 133.

**B.** An IQ of 133 is more unusual because its corresponding $z$-score, $\boxed{2.00}$, is further from 0 than the corresponding $z$-score of $\boxed{-2.13}$ for an IQ of 71.

**C.** Both IQs are equally likely.

Compare the absolute values:

$$|-2.13| = 2.13$$

and

$$|2.00| = 2.00.$$

Since

$$2.13 > 2.00,$$

the IQ of 71 is more unusual.

**Correct Answer: A.**
An IQ of 71 is more unusual because its $z$-score,

$$\boxed{-2.13},$$

is further from 0 than the $z$-score

$$\boxed{2.00}$$

for an IQ of 133.

## 4. Practice Problem 4: Heights of College Females

The dotplot shows heights of college females.
The mean is

$$\bar{x} = 64 \text{ inches}$$

and the standard deviation is

$$s = 3 \text{ inches}.$$

The dotplot is labeled in both inches and standard units:

| Height (inches) | 55 | 58 | 61 | 64 | 67 | 70 | 73 |
|---|---|---|---|---|---|---|---|
| Standard units | $-3$ | $-2$ | $-1$ | 0 | 1 | 2 | 3 |

The number of dots plotted at each height is:

| Height | 55 | 58 | 61 | 64 | 67 | 70 | 73 |
|---|---|---|---|---|---|---|---|
| Number of dots | 1 | 1 | 4 | 15 | 11 | 2 | 0 |

### Part (a): $z$-Score for a Height of 64 Inches

What is the $z$-score for a height of 64 inches (5 feet 4 inches)?
Using

$$z = \frac{x - \bar{x}}{s},$$

we obtain

$$z = \frac{64 - 64}{3} = \frac{0}{3} = \boxed{0}.$$

**Answer:**

$$\boxed{z = 0}$$

A height of 64 inches is exactly equal to the mean height. Therefore, it is 0 standard deviations from the mean.

### Part (b): Height for a $z$-Score of 2

What is the height of a female with a $z$-score of 2?
Start with

$$z = \frac{x - \bar{x}}{s}.$$

Substitute

$$z = 2, \qquad \bar{x} = 64, \qquad s = 3.$$

Thus,

$$2 = \frac{x - 64}{3}.$$

Multiply both sides by 3:

$$6 = x - 64.$$

Add 64:

$$x = 70.$$

Therefore,

$$\boxed{x = 70 \text{ inches}}.$$

**Answer:**

$$\boxed{70 \text{ inches}}$$

A $z$-score of 2 means the height is exactly two standard deviations above the mean:

$$64 + 2(3) = 70 \text{ inches.}$$

## 5. Practice Problem 5: Pollution Indices in Western Cities

In a certain year, a pollution index was calculated for a sample of cities in the western states using data on air and water pollution.

Assume the distribution of pollution indices is unimodal and symmetric.

The mean of the distribution was

$$\bar{x} = 43.3 \text{ points}$$

with a standard deviation of

$$s = 11.5 \text{ points.}$$

### Part (a): Pollution Index Between 31.8 and 54.8

What percentage of western cities would you expect to have a pollution index between 31.8 and 54.8 points?

For the lower endpoint,

$$43.3 - 31.8 = 11.5.$$

For the upper endpoint,

$$54.8 - 43.3 = 11.5.$$

Therefore,

$$31.8 = \bar{x} - s$$

and

$$54.8 = \bar{x} + s.$$

Thus, the interval is within 1 standard deviation of the mean.

By the Empirical Rule, approximately 68% of observations lie within 1 standard deviation of the mean.

**Answer:**

$$\boxed{68\%}$$

### Part (b): Pollution Index Between 20.3 and 66.3

What percentage of western cities would you expect to have a pollution index between 20.3 and 66.3 points?

For the lower endpoint,

$$43.3 - 20.3 = 23.$$

For the upper endpoint,

$$66.3 - 43.3 = 23.$$

Since

$$2(11.5) = 23,$$

the endpoints are exactly two standard deviations below and above the mean:

$$20.3 = \bar{x} - 2s$$

and

$$66.3 = \bar{x} + 2s.$$

By the Empirical Rule, approximately 95% of observations lie within 2 standard deviations of the mean.

**Answer:**

$$\boxed{95\%}$$

### Part (c): Pollution Index of 34.5

The pollution index for one western city in the year was 34.5 points. Based on this distribution, was this unusually high? Explain.
Calculate the $z$-score:

$$z = \frac{x - \bar{x}}{s}.$$

Substituting,

$$z = \frac{34.5 - 43.3}{11.5}.$$

Therefore,

$$z = \frac{-8.8}{11.5} \approx \boxed{-0.77}.$$

The value 34.5 is approximately 0.77 standard deviations below the mean.
It is therefore not unusually high.

### Multiple-Choice Answers

**A.** No, because 34.5 is not more than two standard deviations above the mean.
**B.** Yes, because 34.5 is not more than two standard deviations above the mean.
**C.** No, because 34.5 is more than two standard deviations above the mean.

**D.** Yes, because 34.5 is more than two standard deviations above the mean.

Since

$$z \approx -0.77,$$

the value is actually below the mean and certainly not more than two standard deviations above the mean.

**Correct Answer: A.**
No, because 34.5 is not more than two standard deviations above the mean.

$$\boxed{z \approx -0.77}$$

## 6. Summary: Empirical Rule and $z$-Scores

The Empirical Rule is useful for approximately bell-shaped, unimodal, and symmetric distributions.

**Empirical Rule**
For a bell-shaped distribution:

| Distance from Mean | Approximate Percentage |
|---|---|
| $\bar{x} \pm 1s$ | 68% |
| $\bar{x} \pm 2s$ | 95% |
| $\bar{x} \pm 3s$ | 99.7% |

The $z$-score is calculated using

$$\boxed{z = \frac{x - \bar{x}}{s}}$$

and measures how many standard deviations an observation is from the mean.

- $z > 0$: the observation is above the mean.
- $z < 0$: the observation is below the mean.
- $z = 0$: the observation equals the mean.
- Larger $|z|$: more unusual observation.

When deciding which observation is more unusual, compare the **absolute values** of their $z$-scores.

$$\boxed{\text{More unusual} \quad \Longleftrightarrow \quad \text{larger } |z|}$$