Precalculus

Lecture 1 & 2

Contents

1. Equations and Inequalities

These notes cover the major algebraic techniques required in Precalculus Chapter 1.

Topics include

1. Polynomial inequalities
2. Rational functions
3. Rational equations
4. Rational inequalities
5. Absolute value equations
6. Absolute value inequalities
7. Radical equations
8. Radical inequalities

Throughout these notes we emphasize

- algebraic manipulation,
- factoring,
- graph interpretation,
- interval notation,
- checking for extraneous solutions.

2. Polynomial Inequalities

Polynomial inequalities ask for all values of x that make a polynomial positive, negative, or zero.

Examples include

$$3x - 5 > 0,$$

$$x^2 - 4x + 3 \leq 0,$$

and

$$(x-2)^2(x+1)^3 > 0.$$

Unlike equations, which ask for specific values of x, inequalities ask for intervals of values.

For this reason, understanding the sign of a polynomial is more important than simply finding its zeros.

Polynomial inequalities are divided into three main categories.

1. Linear inequalities
2. Quadratic inequalities
3. General polynomial inequalities

The solution methods become progressively more sophisticated, but all rely on the same basic idea:

Find the critical points, determine the sign on each interval, and select the intervals satisfying the inequality.

2.1 Quadratic Functions

Definition

A quadratic function is any function of the form

$$f(x) = ax^2 + bx + c,$$

where

$$a \neq 0.$$

The graph of every quadratic function is a parabola.

- If $a > 0$, the parabola opens upward.
- If $a < 0$, the parabola opens downward.

The roots (or zeros) of the quadratic are the solutions of

$$ax^2 + bx + c = 0.$$

These roots divide the real number line into intervals where the quadratic is either positive or negative.

Quadratic Formula

When factoring is difficult or impossible, use the quadratic formula.
If

$$ax^2 + bx + c = 0,$$

then

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}.$$

The quantity

$$b^2 - 4ac$$

is called the **discriminant**.

Its value determines the number of real solutions.

Discriminant	Number of Real Roots
> 0	2
$= 0$	1
< 0	0

2.2 Linear Inequalities

A **linear inequality** is an inequality involving a linear expression. Examples include

$$2x + 5 > 9, \qquad 7x - 3 \geq 11, \qquad -4x + 8 < 2.$$

The goal is to determine every value of x that satisfies the inequality.

Unlike solving equations, solving inequalities requires one additional rule:

> **Important Rule**
> Whenever you multiply or divide both sides of an inequality by a negative number, the direction of the inequality reverses.

For example,

$$-2x > 6$$

becomes

$$x < -3,$$

not

$$x > -3.$$

This is one of the most common algebra mistakes, so always check the sign of the coefficient before dividing.

Procedure for Solving Linear Inequalities

1. Simplify both sides if necessary.
2. Move all variable terms to one side.

3. Move all constants to the opposite side.
4. Divide by the coefficient of the variable.
5. Reverse the inequality only if dividing (or multiplying) by a negative number.
6. Express the answer using interval notation whenever possible.

Example 1

Solve

$$7x - 3 \geq 11.$$

Solution

Move the constant to the right side.

$$7x \geq 14.$$

Divide both sides by 7.

$$x \geq 2.$$

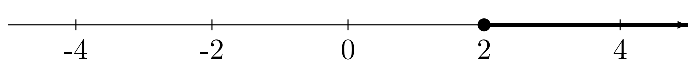
Therefore,

$$\boxed{x \geq 2.}$$

Interval notation:

$$\boxed{[2, \infty).}$$

Number Line Interpretation



The closed circle indicates that $x = 2$ is included.

Example 2

Solve

$$-x + 8 > 3x + 1.$$

Solution

Move the variable terms to one side.

$$-4x > -7.$$

Divide both sides by -4.

Remember to reverse the inequality.

$$x < \frac{7}{4}.$$

Thus,

$$\boxed{x < \frac{7}{4}.}$$

Interval notation:

$$\boxed{(-\infty, \tfrac{7}{4}).}$$

The open circle means the endpoint is not included because the inequality is strict.

Checking Solutions

A quick way to verify your answer is to substitute a number from the solution interval into the original inequality.

For Example 2,

choose $x = 0$.

Original inequality:

$$-0 + 8 > 3(0) + 1$$

which simplifies to

$$8 > 1,$$

which is true.

Now test a number outside the solution interval, such as $x = 2$.

$$-2 + 8 > 6 + 1$$

becomes

$$6 > 7,$$

which is false.

Therefore the solution is correct.

Common Mistakes

- Forgetting to reverse the inequality after dividing by a negative number.
- Forgetting that strict inequalities ($<, >$) use parentheses in interval notation.
- Forgetting that inclusive inequalities ($\leq, \geq$) use brackets.
- Making arithmetic errors while collecting like terms.

2.3 Quadratic Inequalities

Quadratic inequalities involve a quadratic polynomial.

Examples include

$$x^2 - 5x + 6 \leq 0,$$

$$2x^2 + x - 3 > 0,$$

and

$$-3x^2 + 2x + 1 > 0.$$

Unlike linear inequalities, quadratic inequalities usually have multiple solution intervals.

The solutions are determined by studying where the parabola lies above or below the x-axis.

Procedure

1. Move every term to one side.
2. Factor whenever possible.
3. If necessary, use the quadratic formula.
4. Find every zero.
5. Plot the zeros on a number line.
6. Determine the sign on each interval.
7. Choose the intervals satisfying the inequality.

The zeros divide the number line into regions where the quadratic cannot change sign.

Therefore, only one test point is needed from each interval.

Example 1

Solve

$$x^2 - 5x \geq -6.$$

Move every term to one side.

$$x^2 - 5x + 6 \geq 0.$$

Factor.

$$(x-2)(x-3) \geq 0.$$

The critical numbers are

$$x = 2, \qquad x = 3.$$

Test each interval.

Interval	Sign
$(-\infty, 2)$	$+$
$(2, 3)$	$-$
$(3, \infty)$	$+$

Since we want the expression to be nonnegative,

$$\boxed{(-\infty, 2] \cup [3, \infty).}$$

Notice that the endpoints are included because the inequality is ≥ 0.

Example 2

Solve the inequality

$$-3x^2 + 2x + 1 > 0.$$

Step 1: Rewrite the inequality.

Multiply both sides by -1. Since we multiplied by a negative number, reverse the inequality.

$$3x^2 - 2x - 1 < 0.$$

Step 2: Factor the quadratic.

$$3x^2 - 2x - 1 = (3x+1)(x-1).$$

Therefore,

$$(3x+1)(x-1) < 0.$$

The critical numbers are

$$x = -\frac{1}{3}, \qquad x = 1.$$

Step 3: Construct a sign chart.

The real line is divided into three intervals.

$$(-\infty, -\tfrac{1}{3}), \qquad (-\tfrac{1}{3}, 1), \qquad (1, \infty).$$

Test one point from each interval.

Interval	$3x+1$	$x-1$	Product
$(-\infty, -\frac{1}{3})$	$-$	$-$	$+$
$(-\frac{1}{3}, 1)$	$+$	$-$	$-$
$(1, \infty)$	$+$	$+$	$+$

Since we require the expression to be negative,

$$\boxed{\left(-\frac{1}{3}, 1\right)}.$$

Notice that neither endpoint is included because the inequality is strict.

Graphical Interpretation

A quadratic function is represented by a parabola.

The solutions of a quadratic inequality correspond to where the graph lies above or below the x-axis.

- $f(x) > 0$: graph is above the x-axis.
- $f(x) < 0$: graph is below the x-axis.
- $f(x) \geq 0$: above the axis including the intercepts.
- $f(x) \leq 0$: below the axis including the intercepts.

This interpretation is useful even when factoring is difficult.

Summary of Quadratic Inequalities

1. Move every term to one side.
2. Factor whenever possible.
3. Otherwise use the quadratic formula.
4. Find every zero.
5. Draw a sign chart.
6. Test one point from every interval.
7. Include endpoints only for $\leq$ or $\geq$.

2.4 General Polynomial Inequalities

Quadratic inequalities are only one special case of polynomial inequalities.

A polynomial of higher degree behaves similarly:

- Find all zeros.
- Determine the multiplicity of each zero.
- Determine the sign of the polynomial on each interval.

The only new idea is the concept of **multiplicity**.

Multiplicity

Suppose

$$f(x) = (x - a)^m.$$

The exponent m is called the **multiplicity** of the zero $x = a$.

Multiplicity determines how the graph behaves near the zero.

Multiplicity	**Behavior**
Odd	Graph crosses the x-axis
Even	Graph touches and turns around

Equivalently,

- Crossing an odd multiplicity changes the sign.
- Crossing an even multiplicity leaves the sign unchanged.

This fact makes sign charts for higher-degree polynomials very easy.

Procedure for Solving Polynomial Inequalities

1. Move every term to one side.
2. Factor the polynomial completely.
3. Find every real zero.
4. Record the multiplicity of each zero.
5. Place the zeros on a number line.
6. Determine the sign on one interval.
7. Use multiplicity to determine the remaining signs.
8. Select the intervals satisfying the inequality.

Example

Solve

$$-(x+1)^3(5-x)^2(-2x-1)^7 \geq 0.$$

Step 1: Find the zeros.

Each factor equals zero when

$$x=-1, \qquad x=5, \qquad x=-\frac{1}{2}.$$

Their multiplicities are

Zero	Multiplicity
-1	3
$-\frac{1}{2}$	7
5	2

Both multiplicities 3 and 7 are odd, while multiplicity 2 is even.

Step 2: Determine sign changes.

Starting from the far right,

$$x>5,$$

every factor except the leading negative sign is positive.

Therefore the polynomial is negative.

Moving left across each zero,

- crossing $x=5$ (even multiplicity) does not change the sign;

- crossing $x = -\frac{1}{2}$ (odd multiplicity) changes the sign;
- crossing $x = -1$ (odd multiplicity) changes the sign again.

Thus,

Interval	Sign
$(-\infty, -1)$	$-$
$(-1, -\frac{1}{2})$	$+$
$(-\frac{1}{2}, 5)$	$-$
$(5, \infty)$	$-$

Since we require

$$f(x) \geq 0,$$

the solution is

$$\boxed{\left[-1, -\frac{1}{2}\right]}.$$

Notice that the endpoint

$$x = 5$$

is also included because the polynomial equals zero there.

Therefore the complete solution is

$$\boxed{\left[-1, -\frac{1}{2}\right] \cup \{5\}.}$$

Key Ideas

For polynomial inequalities remember:

1. Factor completely.
2. Every real zero splits the number line.
3. Odd multiplicity changes sign.
4. Even multiplicity keeps the same sign.
5. Include endpoints only if equality is allowed.

3. Rational Functions, Equations, and Inequalities

A **rational function** is a function that can be written as the quotient of two polynomials. Rational functions appear throughout precalculus and calculus because they model rates, proportions, and inverse relationships.

Examples include

$$f(x) = \frac{x+1}{x-2},$$

$$g(x) = \frac{x^2-4}{x^3+5},$$

and

$$h(x) = \frac{(x-1)^2(2x+3)^4}{x^2-1}.$$

Unlike polynomial functions, rational functions are not necessarily defined for every real number. The denominator cannot equal zero.

3.1 Definition of a Rational Function

Definition

A rational function has the form

$$f(x) = \frac{p(x)}{q(x)},$$

where

- $p(x)$ and $q(x)$ are polynomials,
- $q(x) \not\equiv 0$.

The numerator and denominator may each have any degree.

Examples include

$$\frac{x^2+2x+1}{x^5-1},$$

and

$$\frac{(x-1)^2(2x+3)^4}{x^2-1}.$$

3.2 Domain of a Rational Function

Every polynomial is defined for all real numbers.

Therefore, the only restrictions on the domain of a rational function occur when the denominator equals zero.

If

$$f(x) = \frac{p(x)}{q(x)},$$

then

$$\boxed{D_f = \{x \in \mathbb{R} : q(x) \neq 0\}.}$$

In words,

The denominator can never equal zero.

Finding the Domain

To determine the domain,

1. Factor the denominator completely.
2. Find every value that makes the denominator zero.
3. Exclude those values.
4. Write the answer in interval notation.

Example 1

Find the domain of

$$f(x) = \frac{(x+1)^2(1-2x)^5}{(x-1)(x+1)(x^2+x+1)}.$$

Step 1

Factor the denominator.

The denominator is already factored:

$$(x-1)(x+1)(x^2+x+1).$$

Step 2

Find where each factor equals zero.

The first factor gives

$$x = 1.$$

The second factor gives

$$x = -1.$$

The quadratic

$$x^2 + x + 1$$

has discriminant

$$1 - 4 = -3 < 0,$$

so it has no real zeros.

Step 3

Exclude the values

$$x = -1, \qquad x = 1.$$

Therefore,

$$\boxed{(-\infty, -1) \cup (-1, 1) \cup (1, \infty).}$$

Important Observation

Notice that the numerator also contains the factor

$$(x + 1)^2.$$

Although one copy of $(x + 1)$ could cancel algebraically, the original function is still undefined at

$$x = -1.$$

Always determine the domain from the original denominator before canceling factors.

3.3 Rational Equations

A rational equation is any equation involving one or more rational expressions.

Examples include

$$\frac{x^2-1}{x+2} = 5,$$

$$\frac{x^2+3x}{x^2-4} = 0,$$

and

$$\frac{1}{x+2} = \frac{2x}{x^2+1}.$$

The presence of denominators introduces the possibility of **extraneous solutions**.

Therefore every answer must be checked.

Method for Solving Rational Equations

When

$$\frac{p(x)}{q(x)} = 0,$$

the denominator does not determine the solution.

Instead,

$$\boxed{\frac{p(x)}{q(x)} = 0 \quad \Longleftrightarrow \quad p(x) = 0}$$

provided that

$$q(x) \neq 0.$$

More complicated rational equations often require multiplying by the least common denominator (LCD) before solving.

General Procedure

1. Determine all domain restrictions.
2. Multiply both sides by the least common denominator if necessary.
3. Solve the resulting polynomial equation.
4. Check every solution in the original equation.
5. Reject any solution that makes a denominator zero.

Example 2

Solve

$$\frac{x^3 + 5x^2 + 6x}{x^5 - 4x^2 + 7} = 0.$$

Since the denominator is not zero,

only the numerator must equal zero.

$$x^3 + 5x^2 + 6x = 0.$$

Factor.

$$x(x^2 + 5x + 6) = 0.$$

Continue factoring.

$$x(x + 2)(x + 3) = 0.$$

Therefore,

$$x = 0, \qquad x = -2, \qquad x = -3.$$

Finally, verify that none of these values make the denominator zero.

Assuming they do not,

$$\boxed{x = -3, \, -2, \, 0.}$$

Example 3

Solve

$$\frac{x^2 - 2x + 1}{x^2 - 1} = 0.$$

Factor both numerator and denominator.

$$\frac{(x - 1)^2}{(x - 1)(x + 1)} = 0.$$

The numerator equals zero only when

$$x = 1.$$

However,

$$x = 1$$

also makes the denominator zero.

Therefore,

$$\boxed{\text{There are no real solutions.}}$$

The value

$$x = 1$$

is called an **extraneous solution** because it does not belong to the domain.

Example 4

Solve

$$\frac{x^2 + 10x + 21}{x + 7} = 0.$$

Step 1: Factor the numerator.

$$x^2 + 10x + 21 = (x + 3)(x + 7).$$

The equation becomes

$$\frac{(x + 3)(x + 7)}{x + 7} = 0.$$

Step 2: Solve the numerator.

A rational expression equals zero only when its numerator is zero.

Thus,

$$(x + 3)(x + 7) = 0.$$

The possible solutions are

$$x = -3, \qquad x = -7.$$

Step 3: Check the domain.

Since the denominator is

$$x + 7,$$

the value

$$x = -7$$

is not in the domain.

Therefore,

$$\boxed{x = -3.}$$

Example 5

Solve

$$\frac{1}{x+2} = \frac{-2x}{x^2+1}.$$

Unlike the previous examples, the equation is not equal to zero, so we cannot simply set the numerator equal to zero.

Instead, eliminate the denominators.

Step 1: Determine the least common denominator.

The denominators are

$$x + 2 \quad \text{and} \quad x^2 + 1.$$

Since they share no common factors,

$$\boxed{\text{LCD} = (x+2)(x^2+1).}$$

Step 2: Multiply every term by the LCD.

$$(x^2 + 1) = -2x(x + 2).$$

Expand the right-hand side.

$$x^2 + 1 = -2x^2 - 4x.$$

Move every term to one side.

$$3x^2 + 4x + 1 = 0.$$

Factor.

$$(3x + 1)(x + 1) = 0.$$

Therefore,

$$x = -\frac{1}{3}, \qquad x = -1.$$

Step 3: Check each solution.

The denominator

$$x + 2$$

is never zero for either solution, and

$$x^2 + 1 > 0$$

for every real number.

Thus both solutions are valid.

$$\boxed{x = -1,\ -\frac{1}{3}.}$$

Common Errors

Many mistakes occur when solving rational equations.

- Forgetting to determine the domain first.
- Canceling factors incorrectly.
- Forgetting to multiply every term by the LCD.
- Not checking for extraneous solutions.

Always substitute your answers back into the original equation.

3.4 Rational Inequalities

A rational inequality asks for all values of x that make a rational expression positive or negative.

Examples include

$$\frac{x-2}{x+1} > 0,$$

$$\frac{x^2-4}{x-3} \le 0,$$

and

$$\frac{1}{x} > 3x+2.$$

Unlike rational equations, rational inequalities require studying the *sign* of the expression on different intervals.

Zeros of the numerator and undefined points of the denominator both divide the real number line into intervals.

Procedure for Solving Rational Inequalities

1. Move every term to one side so that the other side equals zero.
2. Combine all fractions into a single rational expression.
3. Factor the numerator and denominator completely.
4. Find all zeros of the numerator.
5. Find all values that make the denominator zero.
6. Draw a sign chart.
7. Determine the sign of each interval.
8. Include zeros only if equality is allowed.
9. Never include values that make the denominator zero.

Example 1

Solve

$$\frac{x^2-4x+4}{x-5} < 0.$$

Factor the numerator.

$$\frac{(x-2)^2}{x-5} < 0.$$

The critical numbers are

$$x = 2$$

and

$$x = 5.$$

Notice

- $x = 2$ is a zero of multiplicity 2.
- $x = 5$ makes the denominator zero.

The intervals are

$$(-\infty, 2), \qquad (2, 5), \qquad (5, \infty).$$

Since

$$(x - 2)^2 \geq 0$$

for every real number, the numerator is always nonnegative and changes sign at neither side of $x = 2$.

The denominator changes sign at $x = 5$.

Therefore,

Interval	Sign
$(-\infty, 2)$	$-$
$(2, 5)$	$-$
$(5, \infty)$	$+$

Since we want the expression to be negative,

$$\boxed{(-\infty, 2) \cup (2, 5).}$$

Notice that

$$x = 2$$

is excluded because the inequality is strict, and

$$x = 5$$

is excluded because the function is undefined there.

Example 2

Solve

$$\frac{1}{x} > 3x + 2.$$

Move everything to one side.

$$\frac{1}{x} - 3x - 2 > 0.$$

Write as a single fraction.

$$\frac{1 - 3x^2 - 2x}{x} > 0.$$

Factor the numerator.

$$\frac{-(3x-1)(x+1)}{x} > 0.$$

The critical numbers are

$$x = -1, \qquad x = 0, \qquad x = \frac{1}{3}.$$

Here,

- $x = -1$ and $x = \frac{1}{3}$ are zeros,
- $x = 0$ is not in the domain.

The number line is divided into four intervals:

$$(-\infty, -1), \quad (-1, 0), \quad (0, \tfrac{1}{3}), \quad (\tfrac{1}{3}, \infty).$$

Testing one point in each interval gives

Interval	Sign
$(-\infty, -1)$	$+$
$(-1, 0)$	$-$
$(0, \frac{1}{3})$	$+$
$(\frac{1}{3}, \infty)$	$-$

Therefore,

$$\boxed{(-\infty, -1) \cup \left(0, \frac{1}{3}\right).}$$

Example 3

Solve

$$\frac{x^2 + 3x - 2}{x^3} \le 0.$$

Step 1: Factor the numerator.

The quadratic factors as

$$3x^2 + x - 2 = (3x - 2)(x + 1).$$

Thus,

$$\frac{(3x - 2)(x + 1)}{x^3} \le 0.$$

Step 2: Determine the critical numbers.

The numerator is zero when

$$3x - 2 = 0 \quad \Longrightarrow \quad x = \frac{2}{3},$$

or

$$x + 1 = 0 \quad \Longrightarrow \quad x = -1.$$

The denominator is zero when

$$x = 0.$$

Since the denominator contains x^3, the zero at $x = 0$ has multiplicity 3.

The critical numbers are therefore

$$-1, \qquad 0, \qquad \frac{2}{3}.$$

Step 3: Create a sign chart.

These points divide the real line into four intervals.

$$(-\infty, -1), \quad (-1, 0), \quad \left(0, \frac{2}{3}\right), \quad \left(\frac{2}{3}, \infty\right).$$

Testing one point from each interval gives

Interval	Sign
$(-\infty,-1)$	$-$
$(-1,0)$	$+$
$(0,\frac{2}{3})$	$-$
$(\frac{2}{3},\infty)$	$+$

Because the inequality is

$$\leq 0,$$

include intervals where the expression is negative together with the zeros of the numerator.

Hence,

$$\boxed{(-\infty,-1]\cup\left(0,\frac{2}{3}\right].}$$

Notice that $x=0$ is never included because the denominator is zero.

Example 4

Solve

$$\frac{x+1}{-x}<\frac{1}{x-1}-\frac{1}{x^2-1}.$$

This example illustrates how to combine several rational expressions before applying the sign-chart method.

Step 1: Move every term to one side.

Bring all terms to the right-hand side.

$$0<\frac{1}{x-1}-\frac{1}{x^2-1}+\frac{x+1}{x}.$$

Since

$$x^2-1=(x-1)(x+1),$$

the least common denominator is

$$x(x-1)(x+1).$$

Step 2: Combine into a single fraction.

After simplifying,

$$0 < \frac{x^2}{(x-1)(x+1)}.$$

Thus we solve

$$\frac{x^2}{(x-1)(x+1)} > 0.$$

Step 3: Determine the critical numbers.

The numerator equals zero when

$$x = 0,$$

with multiplicity 2.

The denominator equals zero when

$$x = -1, \qquad x = 1.$$

Step 4: Construct the sign chart.

The intervals are

$$(-\infty, -1), \quad (-1, 0), \quad (0, 1), \quad (1, \infty).$$

Because x^2 has even multiplicity, the numerator never changes sign.

The denominator changes sign whenever a simple factor is crossed.

The sign table is

Interval	Sign
$(-\infty, -1)$	$+$
$(-1, 0)$	$-$
$(0, 1)$	$-$
$(1, \infty)$	$+$

Since the inequality is strict,

$$\boxed{(-\infty, -1) \cup (1, \infty).}$$

Common Mistakes

The following mistakes are very common when solving rational inequalities.

1. Forgetting to combine all terms into one rational expression.

2. Forgetting that denominator zeros are *never* included in the solution.
3. Forgetting to include numerator zeros when the inequality contains $\leq$ or $\geq$.
4. Ignoring multiplicity when determining sign changes.
5. Failing to factor completely before making the sign chart.

Summary of Rational Inequalities

For every rational inequality,

1. Rewrite the inequality with zero on one side.
2. Combine everything into one rational function.
3. Factor completely.
4. Find zeros of the numerator.
5. Find values where the denominator is zero.
6. Draw a sign chart.
7. Determine the sign on every interval.
8. Include numerator zeros only if equality is allowed.
9. Never include denominator zeros.

4. The Absolute Value Function

Absolute value measures the distance of a number from zero on the real number line.

Since distance cannot be negative, every absolute value is nonnegative.

Absolute value appears throughout mathematics because it measures distance without regard to direction.

4.1 Definition

The absolute value of a real number x is defined by

$$|x| = \begin{cases} x, & x \geq 0, \\ -x, & x < 0. \end{cases}$$

Thus,

$$|5| = 5,$$

$$|-5| = 5,$$

and

$$|0| = 0.$$

Geometrically,

$$|x|$$

represents the distance between x and the origin.

4.2 Properties of Absolute Value

For any real numbers a and b,

$$|a| \geq 0,$$

with equality only when

$$a = 0.$$

Other important properties include

$$|-a| = |a|,$$

$$|ab| = |a||b|,$$

$$\left|\frac{a}{b}\right| = \frac{|a|}{|b|} \qquad (b \neq 0),$$

$$|a|^2 = a^2,$$

and

$$\sqrt{a^2} = |a|.$$

Finally, the Triangle Inequality states

$$|a + b| \leq |a| + |b|.$$

This inequality is fundamental in higher mathematics and appears frequently in calculus, linear algebra, and analysis.

4.3 Absolute Value Equations

Absolute value equations are solved using the fundamental property of absolute value.

Recall that

$$|x|$$

represents the distance of x from zero.

Since a distance cannot be negative, every absolute value equation must first be checked to ensure that the right-hand side is nonnegative.

Fundamental Theorem

Let a be a real number.

- If $a > 0$,

  $$|x| = a$$

  if and only if

  $$x = a \quad \text{or} \quad x = -a.$$

- If $a = 0$,

  $$|x| = 0$$

  if and only if

  $$x = 0.$$

- If $a < 0$,

  $$|x| = a$$

  has no real solution.

More generally,

$$|A| = |B|$$

if and only if

$$A = B$$

or

$$A = -B.$$

These two cases form the basis of every absolute value equation.

Procedure

To solve an absolute value equation,

1. Isolate the absolute value expression.
2. Check whether the opposite side is negative.
3. If it is negative, there are no real solutions.
4. Otherwise, write two separate equations.
5. Solve each equation.
6. Check every answer in the original equation.

Example 1

Solve

$$|4x - 3| = 12.$$

Since the right-hand side is positive, there are two possible cases.

$$4x - 3 = 12$$

or

$$4x - 3 = -12.$$

First equation:

$$4x = 15,$$

so

$$x = \frac{15}{4}.$$

Second equation:

$$4x = -9,$$

so

$$x = -\frac{9}{4}.$$

Therefore,

$$\boxed{x = \frac{15}{4} \quad \text{or} \quad x = -\frac{9}{4}.}$$

Example 2

Solve

$$|x - 1| = |x + 2|.$$

Using the theorem,

$$x - 1 = x + 2$$

or

$$x - 1 = -(x + 2).$$

The first equation becomes

$$-1 = 2,$$

which is impossible.

Therefore it gives no solution.

The second equation becomes

$$2x = -1,$$

so

$$x = -\frac{1}{2}.$$

Checking,

$$\left|-\frac{1}{2} - 1\right| = \frac{3}{2},$$

and

$$\left|-\frac{1}{2}+2\right| = \frac{3}{2}.$$

Both sides agree.

Hence,

$$\boxed{x = -\frac{1}{2}.}$$

Example 3

Solve

$$|x^2 - 2x| = 3x - 6.$$

First observe that

$$3x - 6$$

must be nonnegative.

Thus,

$$x \geq 2.$$

Now consider the two cases.

$$x^2 - 2x = 3x - 6$$

or

$$x^2 - 2x = -(3x - 6).$$

First equation:

$$x^2 - 5x + 6 = 0.$$

Factor.

$$(x - 2)(x - 3) = 0.$$

Hence,

$$x = 2 \quad \text{or} \quad x = 3.$$

Second equation:

$$x^2 + x - 6 = 0.$$

Factor.

$$(x + 3)(x - 2) = 0.$$

Possible solutions are

$$x = -3, \qquad x = 2.$$

Now check every candidate.

$$x = -3$$

does not satisfy the original equation because the right-hand side is negative.

Both

$$x = 2$$

and

$$x = 3$$

work.

Therefore,

$$\boxed{x = 2,\ 3.}$$

4.4 Absolute Value Inequalities

Absolute value inequalities are solved using two fundamental rules.

Unlike equations, inequalities produce intervals of solutions.

Fundamental Rules

Suppose

$$a > 0.$$

Then

$$|x| < a$$

is equivalent to

$$-a < x < a.$$

In words,

The variable must stay within the interval.

Likewise,

$$|x| > a$$

is equivalent to

$$x < -a \quad \text{or} \quad x > a.$$

In words,

The variable must lie outside the interval.

These rules extend naturally to

$$\leq \quad \text{and} \quad \geq .$$

Specifically,

$$|x| \leq a$$

becomes

$$-a \leq x \leq a,$$

while

$$|x| \geq a$$

becomes

$$x \leq -a \quad \text{or} \quad x \geq a.$$

Important Observations

Before solving any absolute value inequality, check the number on the right-hand side.

- If

$$|A| < 0,$$

there are no solutions.

- If

$$|A| \geq 0,$$

every real number satisfies the inequality.

- Always isolate the absolute value before applying the theorem.

Example 1

Solve

$$|-2x - 1| \geq 0.$$

Solution

By definition, the absolute value of every real number is nonnegative.

Therefore,

$$|A| \geq 0$$

is true for every real value of A.

Hence,

$$\boxed{\text{All real numbers satisfy the inequality.}}$$

In interval notation,

$$\boxed{(-\infty, \infty).}$$

Example 2

Solve

$$|4x - 5| < -2.$$

Solution

The left-hand side is always greater than or equal to zero.

Since no nonnegative number can be less than a negative number,

$$|4x - 5| < -2$$

has no real solution.

Thus,

$$\boxed{\varnothing.}$$

Whenever the right-hand side of an absolute value inequality is negative,

$$|A| < c, \qquad c < 0,$$

there are no solutions.

Example 3

Solve

$$|2 - 3x| < 3.$$

Since the inequality has the form

$$|A| < a,$$

we write

$$-3 < 2 - 3x < 3.$$

Now solve each inequality.

First inequality:

$$-3 < 2 - 3x.$$

Subtract 2.

$$-5 < -3x.$$

Divide by -3, reversing the inequality.

$$x < \frac{5}{3}.$$

Second inequality:

$$2 - 3x < 3.$$

Subtract 2.

$$-3x < 1.$$

Divide by -3.

$$x > -\frac{1}{3}.$$

Combining both conditions gives

$$-\frac{1}{3} < x < \frac{5}{3}.$$

Therefore,

$$\boxed{\left(-\frac{1}{3}, \frac{5}{3}\right).}$$

Example 4

Solve

$$|5x + 1| \geq 11.$$

Since the inequality has the form

$$|A| \geq a,$$

there are two separate cases.

$$5x + 1 \leq -11$$

or

$$5x + 1 \geq 11.$$

Case 1:

$$5x \leq -12,$$

so

$$x \leq -\frac{12}{5}.$$

Case 2:

$$5x \geq 10,$$

so

$$x \geq 2.$$

Hence,

$$\boxed{(-\infty, -\tfrac{12}{5}] \cup [2, \infty).}$$

Summary of Absolute Value Inequalities

The following table summarizes the four most common cases.

Absolute Value Inequality	**Equivalent Statement**
$\lvert A\rvert < a$	$-a < A < a$
$\lvert A\rvert \leq a$	$-a \leq A \leq a$
$\lvert A\rvert > a$	$A < -a$ or $A > a$
$\lvert A\rvert \geq a$	$A \leq -a$ or $A \geq a$

Always isolate the absolute value before applying one of these four rules.

5. Radical Equations and Inequalities

Radical equations contain variables inside radicals.

Examples include

$$\sqrt{x+2} = x - 4,$$

$$\sqrt[3]{2x-5} + 2 = x,$$

and

$$x^{3/2} = 64.$$

Unlike polynomial equations, radical equations often produce **extraneous solutions** after raising both sides to a power.

Therefore every proposed solution must always be checked in the original equation.

5.1 Equations Involving Only Radicals

The general strategy is to isolate one radical, eliminate it by raising both sides to the appropriate power, and repeat until no radicals remain.

Procedure

1. Isolate one radical.
2. Raise both sides to the power that removes that radical.
3. Simplify.
4. If radicals remain, isolate another radical and repeat.
5. Solve the resulting polynomial equation.
6. Check every solution in the original equation.

Example 1

Solve

$$\sqrt{x+2} + 4 - x = 0.$$

Move the non-radical terms to the opposite side.

$$\sqrt{x+2} = x - 4.$$

Square both sides.

$$x + 2 = (x-4)^2.$$

Expand.

$$x + 2 = x^2 - 8x + 16.$$

Move everything to one side.

$$x^2 - 9x + 14 = 0.$$

Factor.

$$(x-2)(x-7)=0.$$

The possible solutions are

$$x=2, \qquad x=7.$$

Check both values.

For

$$x=2,$$

the original equation becomes

$$\sqrt{4}+4-2=2+2=4\neq 0.$$

Thus $x=2$ is **not** a solution.

For

$$x=7,$$

$$\sqrt{9}+4-7=3+4-7=0.$$

Hence,

$$\boxed{x=7.}$$

Remark. The lecture slide states that only $x=2$ works. This is a typographical error. Direct substitution shows that $\boxed{x=7}$ is the correct solution.

Example 2

Solve

$$\sqrt[3]{2x-5}+2=x.$$

Step 1: Isolate the cube root.

$$\sqrt[3]{2x-5}=x-2.$$

Step 2: Cube both sides.

$$2x-5=(x-2)^3.$$

Expand the right-hand side:

$$(x-2)^3 = x^3 - 6x^2 + 12x - 8.$$

So,

$$2x - 5 = x^3 - 6x^2 + 12x - 8.$$

Step 3: Move everything to one side.

$$0 = x^3 - 6x^2 + 10x - 3.$$

Step 4: Factor.

Test rational roots. $x = 3$ works:

$$0 = (x-3)(x^2 - 3x + 1).$$

So the solutions are:

$$x = 3, \qquad x = \frac{3 \pm \sqrt{5}}{2}.$$

Step 5: Check in original equation.

All three solutions satisfy the original equation.

Thus,

$$\boxed{x = 3,\ \frac{3+\sqrt{5}}{2},\ \frac{3-\sqrt{5}}{2}.}$$

5.2 Radical Equations with Powers

Some equations involve both powers and radicals. A common strategy is to first simplify exponents or rewrite radicals using rational exponents.

Recall:

$$\sqrt[n]{x^m} = x^{m/n}.$$

Example 1

Solve

$$x^{3/2} = 64.$$

Step 1: Rewrite in radical form.

$$x^{3/2} = (\sqrt{x})^3 = 64.$$

Step 2: Take square root first.

$$\sqrt{x} = 4.$$

Step 3: Square both sides.

$$x = 16.$$

Thus,

$$\boxed{x = 16.}$$

Example 2

Solve

$$x^{2/5} = -1.$$

Since the exponent has an even root in the denominator,

$$x^{2/5} = (\sqrt[5]{x})^2 \geq 0$$

for all real x.

Therefore it cannot equal a negative number.

$$\boxed{\text{No real solution.}}$$

Example 3

Solve

$$(6x-3)^{5/3} = -243.$$

Step 1: Rewrite.

$$(6x-3)^{5/3} = \left(\sqrt[3]{6x-3}\right)^5.$$

Step 2: Take the fifth root.

$$\sqrt[3]{6x-3} = -3.$$

Step 3: Cube both sides.

$$6x - 3 = -27.$$

Step 4: Solve.

$$6x = -24, \qquad x = -4.$$

Thus,

$$\boxed{x = -4.}$$

Example 4

Solve

$$(2x+5)^{7/2} = -1.$$

Since

$$(2x+5)^{7/2} = \left(\sqrt{2x+5}\right)^7 \geq 0$$

for all real x, the expression cannot be negative.

Thus,

$$\boxed{\text{No real solution.}}$$

5.3 Radical Inequalities

Radical inequalities are solved using sign analysis, similar to polynomial and rational inequalities. However, we must also consider domain restrictions imposed by even roots.

Procedure

1. Move all terms to one side.
2. Identify domain restrictions (especially even roots).
3. Factor where possible.
4. Find critical points (zeros and undefined points).
5. Construct a sign chart.
6. Check domain validity on each interval.

Example 1

Solve

$$(1-x)^3(x-3)^4 > 0.$$

Step 1: Identify zeros.

$$x = 1, \qquad x = 3.$$

Multiplicities:

$$x = 1\ (3), \qquad x = 3\ (4).$$

Step 2: Sign analysis.

- Odd multiplicity at $x = 1 \to$ sign changes - Even multiplicity at $x = 3 \to$ no sign change

Step 3: Test intervals.

$$(-\infty, 1), \quad (1, 3), \quad (3, \infty).$$

Resulting signs:

$$(-, +, -).$$

We want > 0, so only

$$\boxed{(-\infty, 1)}.$$

Example 2

Solve

$$(x-1)^3(x+2)^5(2-x)^3 \geq 0.$$

Step 1: Zeros.

$$x = 1, \quad x = -2, \quad x = 2.$$

Step 2: Multiplicities.

All are odd $\to$ sign changes at each zero.

Step 3: Order the zeros.

$$-2, \quad 1, \quad 2.$$

Step 4: Sign chart.

Starting from the right:

$$(2, \infty) : +, \quad (1, 2) : -, \quad (-2, 1) : +, \quad (-\infty, -2) : -.$$

Step 5: Include zeros (since ≥ 0).

$$\boxed{[-2, 1] \cup [2, \infty).}$$