Precalculus

Lecture 5

Contents

---

1. Quadratic Functions and Their Graphs

A **quadratic function** is a polynomial function of degree two. Its general form is

$$\boxed{f(x) = ax^2 + bx + c, \qquad a \neq 0.}$$

The graph of every quadratic function is a curve called a **parabola**.
The coefficient a determines the direction in which the parabola opens:

$$\begin{cases} a > 0 & \text{the parabola opens upward,} \\ a < 0 & \text{the parabola opens downward.} \end{cases}$$

When the parabola opens upward, its vertex is its lowest point. When it opens downward, its vertex is its highest point.

Forms of a Quadratic Function. Quadratic functions commonly appear in three forms.

General form..

$$\boxed{f(x) = ax^2 + bx + c.}$$

This form makes the leading coefficient and y-intercept easy to identify.
Since

$$f(0) = c,$$

the y-intercept is

$$\boxed{(0, c).}$$

Vertex form..

$$\boxed{f(x) = a(x-h)^2 + k.}$$

In this form, the vertex is immediately visible:

$$\boxed{(h, k).}$$

The vertical line

$$\boxed{x = h}$$

is the axis of symmetry.

Factored form..

$$\boxed{f(x) = a(x - r_1)(x - r_2).}$$

When the quadratic has real roots, this form displays its x-intercepts:

$$\boxed{(r_1, 0) \quad \text{and} \quad (r_2, 0).}$$

1.1 The Vertex of a Quadratic Function

Every parabola has exactly one vertex.
For a quadratic function in general form,

$$f(x) = ax^2 + bx + c,$$

the x-coordinate of the vertex is

$$\boxed{x_v = -\frac{b}{2a}.}$$

The corresponding y-coordinate is

$$y_v = f\left(-\frac{b}{2a}\right).$$

Therefore, the vertex is

$$\boxed{\left(-\frac{b}{2a}, f\left(-\frac{b}{2a}\right)\right).}$$

The axis of symmetry is

$$\boxed{x = -\frac{b}{2a}.}$$

Maximum and Minimum Values. The vertex determines the absolute maximum or minimum value of a quadratic function.
If

$$a > 0,$$

the parabola opens upward, and the vertex gives the absolute minimum.
If

$$a < 0,$$

the parabola opens downward, and the vertex gives the absolute maximum.
Thus,

$a > 0$	absolute minimum at the vertex
$a < 0$	absolute maximum at the vertex

Finding the Intercepts.

y-intercept.. Set

$$x = 0.$$

The point is

$$\boxed{(0, f(0)).}$$

x-intercepts.. Set

$$f(x) = 0$$

and solve the resulting quadratic equation.
Possible methods include

- factoring,
- the square-root property,
- completing the square,

- the quadratic formula.

The quadratic formula is

$$\boxed{x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}.}$$

The discriminant

$$\Delta = b^2 - 4ac$$

determines the number of real x-intercepts:

$\Delta > 0$	two distinct real intercepts
$\Delta = 0$	one repeated real intercept
$\Delta < 0$	no real intercepts

Example 1: A Quadratic in Vertex Form. Find the vertex and the x- and y-intercepts of

$$f(x) = 4(x-1)^2 - 3.$$

Vertex.. The function is already in vertex form

$$f(x) = a(x-h)^2 + k.$$

Therefore,

$$h = 1, \qquad k = -3,$$

so the vertex is

$$\boxed{(1, -3).}$$

The axis of symmetry is

$$\boxed{x = 1.}$$

Since

$$a = 4 > 0,$$

the parabola opens upward, and the minimum value is

$$\boxed{-3.}$$

y-intercept.. Set $x = 0$:

$$\begin{aligned} f(0) &= 4(0-1)^2 - 3 \\ &= 4 - 3 \\ &= 1. \end{aligned}$$

Thus, the y-intercept is

$$\boxed{(0, 1).}$$

x-intercepts.. Set $f(x) = 0$:

$$4(x-1)^2 - 3 = 0.$$

Then

$$4(x-1)^2 = 3,$$

so

$$(x-1)^2 = \frac{3}{4}.$$

Taking square roots,

$$x - 1 = \pm\frac{\sqrt{3}}{2}.$$

Therefore,

$$x = 1 \pm \frac{\sqrt{3}}{2}.$$

The x-intercepts are

$$\boxed{\left(1 - \frac{\sqrt{3}}{2}, 0\right) \quad \text{and} \quad \left(1 + \frac{\sqrt{3}}{2}, 0\right).}$$

Example 2: A Quadratic in General Form. Find the vertex and the x- and y-intercepts of

$$f(x) = -3x^2 + 6x - 2.$$

Vertex.. Here,

$$a = -3, \qquad b = 6, \qquad c = -2.$$

The x-coordinate of the vertex is

$$x_v = -\frac{b}{2a} = -\frac{6}{2(-3)} = 1.$$

Now evaluate $f(1)$:

$$\begin{aligned} f(1) &= -3(1)^2 + 6(1) - 2 \\ &= 1. \end{aligned}$$

Therefore, the vertex is

$$\boxed{(1, 1).}$$

Since $a < 0$, the parabola opens downward, and the absolute maximum value is

$$\boxed{1.}$$

The axis of symmetry is

$$\boxed{x = 1.}$$

y-intercept..

$$f(0) = -2.$$

Thus,

$$\boxed{(0, -2)}$$

is the y-intercept.

x-intercepts.. Solve

$$-3x^2 + 6x - 2 = 0.$$

Using the quadratic formula,

$$x = \frac{-6 \pm \sqrt{6^2 - 4(-3)(-2)}}{2(-3)}.$$

Simplify:

$$\begin{aligned} x &= \frac{-6 \pm \sqrt{36 - 24}}{-6} \\ &= \frac{-6 \pm \sqrt{12}}{-6} \\ &= \frac{-6 \pm 2\sqrt{3}}{-6}. \end{aligned}$$

Therefore,

$$x = 1 \mp \frac{\sqrt{3}}{3}.$$

Equivalently,

$$x = \frac{3 - \sqrt{3}}{3} \qquad \text{or} \qquad x = \frac{3 + \sqrt{3}}{3}.$$

Hence, the x-intercepts are

$$\boxed{\left(\frac{3 - \sqrt{3}}{3}, 0\right) \quad \text{and} \quad \left(\frac{3 + \sqrt{3}}{3}, 0\right).}$$

2. General Polynomial Functions

In the previous section we studied quadratic functions, which are polynomials of degree two. We now extend these ideas to polynomial functions of arbitrary degree.

Polynomial functions are among the most important functions in mathematics. They arise naturally in algebra, calculus, engineering, economics, computer science, and many physical sciences. Their graphs exhibit a wide variety of shapes depending on their degree and leading coefficient.

2.1 Definition of a Polynomial Function

A polynomial function of degree n has the general form

$$\boxed{P(x) = a_n x^n + a_{n-1} x^{n-1} + \cdots + a_2 x^2 + a_1 x + a_0,}$$

where

$$a_n \neq 0,$$

and all exponents are nonnegative integers.
The number

$$n$$

is called the **degree** of the polynomial.
The coefficients

$$a_0, a_1, \ldots, a_n$$

are real numbers.

Important Terminology. For the polynomial

$$P(x) = a_n x^n + a_{n-1} x^{n-1} + \cdots + a_0,$$

we define

- a_n is the **leading coefficient**;
- $a_n x^n$ is the **leading term**;
- a_0 is the **constant term**;
- n is the **degree**.

Examples.

$$P(x) = 3x^5 - 2x^3 + x - 7$$

is a polynomial of degree

$$5.$$

Its leading coefficient is

$$3,$$

and its constant term is

$$-7.$$

$$Q(x) = 7x^8 + 4x^2 + 1$$

has degree

$$8.$$

$$R(x) = 6$$

is called a constant polynomial and has degree zero.

What Is Not a Polynomial?. The following are **not** polynomial functions.

$$\frac{1}{x}$$

because it contains

$$x^{-1}.$$

$$\sqrt{x} = x^{1/2}$$

because it contains a fractional exponent.

$$2^x$$

because the variable appears in the exponent.

$$\frac{x+1}{x-2}$$

because the variable appears in the denominator.

2.2 Zeros of a Polynomial

A number

$$c$$

is called a
zero (or root)
of a polynomial if

$$\boxed{P(c) = 0.}$$

Every zero corresponds to an

$$x\text{-intercept}$$

of the graph.
For example,

$$P(x) = (x-2)(x+3)$$

has zeros

$$\boxed{x = 2 \quad \text{and} \quad x = -3.}$$

2.3 The Zero Product Property

The following theorem is one of the most useful tools for solving polynomial equations.

Theorem 2.1 (Zero Product Property)**.** *If*

$$AB = 0,$$

then

$$\boxed{A = 0 \qquad or \qquad B = 0.}$$

This theorem allows polynomial equations to be solved after they have been factored.

Example 1. Solve

$$x^5 + 7x^4 + 12x^3 = 0.$$

Solution. Factor out the greatest common factor.

$$x^5 + 7x^4 + 12x^3 = x^3(x^2 + 7x + 12).$$

Now factor the quadratic.

$$x^2 + 7x + 12 = (x + 3)(x + 4).$$

Hence,

$$x^3(x + 3)(x + 4) = 0.$$

Applying the Zero Product Property,

$$x = 0, \qquad x = -3, \qquad x = -4.$$

Therefore,

$$\boxed{x = 0,\ -3,\ -4.}$$

Notice that

$$x = 0$$

is a zero of multiplicity

$$3.$$

Example 2. Solve

$$2x^4 - 5x^2 = 25.$$

Solution. Move all terms to one side.

$$2x^4 - 5x^2 - 25 = 0.$$

Factor.

$$(2x^2 + 5)(x^2 - 5) = 0.$$

The first factor,

$$2x^2 + 5 = 0,$$

has no real solutions because

$$x^2 = -\frac{5}{2}.$$

The second factor gives

$$x^2 = 5,$$

so

$$\boxed{x = \pm\sqrt{5}.}$$

These are the only real solutions.

The Number of Solutions. One of the fundamental results in algebra is the following.

Theorem 2.2. *A polynomial equation of degree*

$$n$$

has at most

$$\boxed{n}$$

real solutions.

Moreover, the Fundamental Theorem of Algebra states that every polynomial of degree n has exactly n complex zeros when multiplicities are counted.

For example,

$$x^4 - 1 = 0$$

has four complex zeros,
although only two are real.

3. The Rational Root Theorem

Factoring higher-degree polynomials is often much more difficult than factoring quadratic polynomials. In many situations there is no obvious common factor, and simple factoring techniques do not apply.

Fortunately, the Rational Root Theorem provides a systematic way of finding possible rational zeros of a polynomial with integer coefficients.

Once one rational zero has been found, polynomial division may be used to reduce the degree of the polynomial, making the remaining factorization much easier.

Statement of the Rational Root Theorem. Consider the polynomial

$$P(x) = a_n x^n + a_{n-1}x^{n-1} + \cdots + a_1 x + a_0,$$

where every coefficient is an integer.

Theorem 3.1 (Rational Root Theorem)**.** *If*

$$\frac{p}{q}$$

is a rational zero of

$$P(x),$$

written in lowest terms, then

$$\boxed{p \text{ divides } a_0, \qquad q \text{ divides } a_n.}$$

Consequently, every possible rational zero must have the form

$$\boxed{\pm\frac{\textit{factor of } a_0}{\textit{factor of } a_n}.}$$

Notice that the theorem does not tell us which numbers are actually roots.
Instead, it produces a finite list of candidates that must be tested.

Procedure for Applying the Rational Root Theorem. To use the Rational Root Theorem effectively, follow these steps.

1. Identify the leading coefficient
$$a_n.$$
2. Identify the constant term
$$a_0.$$
3. List every factor of
$$a_0.$$
4. List every factor of
$$a_n.$$
5. Form every possible fraction
$$\pm\frac{\text{factor of } a_0}{\text{factor of } a_n}.$$
6. Test each candidate by substitution or synthetic division.

Once a root has been found, the polynomial can be divided by its corresponding factor.

Factor Theorem. The Rational Root Theorem is closely related to the following important result.

Theorem 3.2 (Factor Theorem)**.** *The number*

$$c$$

is a zero of

$$P(x)$$

if and only if

$$\boxed{x-c}$$

is a factor of

$$P(x).$$

Equivalently,

$$P(c) = 0$$

if and only if

$$P(x) = (x-c)Q(x)$$

for some polynomial

$$Q(x).$$

Therefore, once a root has been discovered, we immediately know one factor of the polynomial.

Example. Factor the polynomial

$$P(x) = x^3 + 3x^2 - 20.$$

Step 1. Identify the Leading and Constant Coefficients. Here,

$$a_3 = 1, \qquad a_0 = -20.$$

Step 2. List the Possible Rational Roots. Since the leading coefficient is

$$1,$$

the denominator of every rational root must be

$$1.$$

The factors of

$$20$$

are

$$1,\ 2,\ 4,\ 5,\ 10,\ 20.$$

Hence the possible rational roots are

$$\boxed{\pm 1, \pm 2, \pm 4, \pm 5, \pm 10, \pm 20.}$$

Step 3. Test the Candidates. Evaluate the polynomial at each candidate.
First,

$$P(1) = 1 + 3 - 20 = -16,$$

so

$$1$$

is not a root.
Next,

$$\begin{aligned} P(2) &= 2^3 + 3(2)^2 - 20 \\ &= 8 + 12 - 20 \\ &= 0. \end{aligned}$$

Therefore,

$$\boxed{x = 2}$$

is a root.

Step 4. Apply the Factor Theorem. Since

$$P(2) = 0,$$

the polynomial contains the factor

$$x - 2.$$

Dividing

$$x^3 + 3x^2 - 20$$

by

$$x - 2$$

gives

$$x^2 + 5x + 10.$$

Hence,

$$\boxed{x^3 + 3x^2 - 20 = (x - 2)(x^2 + 5x + 10).}$$

Step 5. Determine Whether Further Factoring Is Possible. Consider the quadratic

$$x^2 + 5x + 10.$$

Its discriminant is

$$\Delta = 5^2 - 4(1)(10) = 25 - 40 = -15.$$

Since the discriminant is negative, the quadratic has no real roots and cannot be factored over the real numbers.
Therefore,

$$\boxed{x^3 + 3x^2 - 20 = (x - 2)(x^2 + 5x + 10)}$$

is the complete factorization over the real numbers.

Remarks. The Rational Root Theorem greatly reduces the amount of trial and error needed when factoring polynomials.
Without the theorem, infinitely many rational numbers could potentially be tested.
Instead, the theorem guarantees that only finitely many candidates need to be checked.
For higher-degree polynomials, synthetic division is often used together with the Rational Root Theorem to test each possible root efficiently.

4. Polynomial Division

Polynomial division is the algebraic analogue of ordinary long division with numbers. It is an essential tool for factoring higher-degree polynomials, finding zeros of polynomial functions, simplifying rational expressions, and performing partial fraction decomposition.
Whenever a polynomial is divided by another polynomial, there exist unique polynomials called the *quotient* and the *remainder* satisfying

$$\boxed{P(x) = D(x)Q(x) + R(x),}$$

where

$$\deg(R) < \deg(D).$$

Here,

- $P(x)$ is the dividend,
- $D(x)$ is the divisor,
- $Q(x)$ is the quotient,
- $R(x)$ is the remainder.

Consequently,

$$\boxed{\frac{P(x)}{D(x)} = Q(x) + \frac{R(x)}{D(x)}.}$$

Why Polynomial Division Is Important. Polynomial division has many important applications. It is used to

- factor higher-degree polynomials,
- verify whether a number is a zero of a polynomial,
- determine the quotient after one factor has been found,
- simplify rational expressions,
- perform partial fraction decomposition,
- determine oblique (slant) asymptotes of rational functions.

Long Division Algorithm. The long division algorithm closely resembles the familiar long division of integers.

The procedure is summarized below.

1. Divide the leading term of the dividend by the leading term of the divisor.
2. Write the result as the next term of the quotient.
3. Multiply the divisor by this term.
4. Subtract the result from the current dividend.
5. Bring down the next term.
6. Repeat until the degree of the remainder is less than the degree of the divisor.

Example. Divide

$$2x^4 - 10x^3 + 11x^2 + 6x + 2$$

by

$$x - 3.$$

Step 1. Divide the Leading Terms. The leading terms are

$$2x^4 \quad \text{and} \quad x.$$

Therefore,

$$\frac{2x^4}{x} = 2x^3.$$

Write

$$2x^3$$

as the first term of the quotient.

Step 2. Multiply. Multiply the divisor by

$$2x^3.$$

$$2x^3(x-3) = 2x^4 - 6x^3.$$

Step 3. Subtract. Subtract the product from the dividend.

$$\begin{aligned}&(2x^4 - 10x^3 + 11x^2 + 6x + 2)\\ &\quad - (2x^4 - 6x^3)\\ &= -4x^3 + 11x^2 + 6x + 2.\end{aligned}$$

Step 4. Repeat. Divide

$$-4x^3$$

by

$$x.$$

This gives

$$-4x^2.$$

Place

$$-4x^2$$

in the quotient.
Multiply:

$$-4x^2(x-3) = -4x^3 + 12x^2.$$

Subtract:

$$\begin{aligned}&(-4x^3 + 11x^2 + 6x + 2)\\ &\quad - (-4x^3 + 12x^2)\\ &= -x^2 + 6x + 2.\end{aligned}$$

Step 5. Continue the Process. Divide

$$-x^2$$

by

$$x,$$

giving

$$-x.$$

Multiply:

$$-x(x-3) = -x^2 + 3x.$$

Subtract:

$$\begin{aligned}&(-x^2 + 6x + 2)\\ &\qquad - (-x^2 + 3x)\\ &= 3x + 2.\end{aligned}$$

Step 6. Final Division. Divide

$$3x$$

by

$$x,$$

obtaining

$$3.$$

Multiply:

$$3(x-3) = 3x - 9.$$

Subtract:

$$(3x+2) - (3x-9) = 11.$$

Since the remainder is a constant, the division is complete.

Final Answer. The quotient is

$$2x^3 - 4x^2 - x + 3,$$

and the remainder is

$$11.$$

Hence,

$$\boxed{\frac{2x^4 - 10x^3 + 11x^2 + 6x + 2}{x-3} = 2x^3 - 4x^2 - x + 3 + \frac{11}{x-3}.}$$

The Remainder Theorem. Polynomial division leads directly to another important theorem.

Theorem 4.1 (Remainder Theorem)**.** *If a polynomial*

$$P(x)$$

is divided by

$$x - c,$$

then the remainder is

$$\boxed{P(c).}$$

Therefore,

$$P(c) = 0$$

if and only if

$$x - c$$

is a factor of the polynomial.

This theorem provides a quick way to test whether a number is a zero without performing the entire division.

Synthetic Division. When the divisor has the form

$$x - c,$$

synthetic division provides a faster alternative to long division.

Instead of writing every power of x, synthetic division uses only the coefficients of the polynomial.

Although the underlying arithmetic is identical to long division, synthetic division is usually much quicker and is the preferred method when testing possible rational roots obtained from the Rational Root Theorem.

5. Power Functions and End Behavior

Before studying the graphs of general polynomial functions, it is helpful to understand the behavior of the simpler functions

$$f(x) = x^n,$$

called **power functions**.

These functions form the building blocks of polynomial functions. In fact, the graph of any polynomial behaves like its highest-degree term for very large positive and negative values of x.

Understanding power functions therefore allows us to predict the end behavior of all polynomial graphs.

Power Functions. A power function is any function of the form

$$\boxed{f(x) = x^n,}$$

where

$$n$$

is a positive integer.

The shape of the graph depends entirely on whether the exponent is even or odd.

Even Power Functions. Consider

$$f(x) = x^2, \qquad x^4, \qquad x^6, \qquad \ldots$$

These functions all share the same important properties.

Symmetry. Since

$$(-x)^n = x^n,$$

whenever

$$n$$

is even,

$$\boxed{f(-x) = f(x).}$$

Therefore, every even power function is symmetric about the y**-axis**.

Domain.

$$\boxed{(-\infty, \infty)}$$

Range.

$$\boxed{[0, \infty).}$$

Intercept. Every even power function passes through

$$\boxed{(0, 0).}$$

End Behavior. As

$$x \to \infty,$$

$$x^n \to \infty.$$

As

$$x \to -\infty,$$

$$x^n \to \infty.$$

Thus,

$$\boxed{\begin{aligned} x \to \infty &\Longrightarrow f(x) \to \infty, \\ x \to -\infty &\Longrightarrow f(x) \to \infty. \end{aligned}}$$

Both ends of the graph point upward.

Odd Power Functions. Now consider

$$x, \qquad x^3, \qquad x^5, \qquad \ldots$$

Again, these functions have several common properties.

Symmetry. For odd integers,

$$(-x)^n = -x^n.$$

Hence,

$$\boxed{f(-x) = -f(x).}$$

Such functions are symmetric about the

$$\boxed{\text{origin.}}$$

Domain.

$$\boxed{(-\infty, \infty).}$$

Range.

$$\boxed{(-\infty, \infty).}$$

Intercept. Every odd power function passes through

$$\boxed{(0, 0).}$$

End Behavior. As

$$x \to \infty,$$

$$x^n \to \infty.$$

As

$$x \to -\infty,$$

$$x^n \to -\infty.$$

Therefore,

$$\boxed{\begin{gathered} x \to \infty \Longrightarrow f(x) \to \infty, \\ x \to -\infty \Longrightarrow f(x) \to -\infty. \end{gathered}}$$

The graph falls to the left and rises to the right.

Leading Term Determines End Behavior. One of the most important facts about polynomial functions is that the highest-degree term dominates the polynomial for large values of $|x|$.
Consider the polynomial

$$P(x) = a_n x^n + a_{n-1}x^{n-1} + \cdots + a_0.$$

When

$$|x|$$

becomes very large,
the lower-degree terms become insignificant compared to

$$a_n x^n.$$

Consequently,

$$\boxed{P(x) \approx a_n x^n \quad \text{for large } |x|.}$$

Therefore,
the graph of the polynomial has the same end behavior as the graph of its leading term.

The Four Possible Cases. The degree of the polynomial and the sign of its leading coefficient completely determine its end behavior.

Degree	**Leading Coefficient**	**End Behavior**
Even	$a_n > 0$	Up on both ends
Even	$a_n < 0$	Down on both ends
Odd	$a_n > 0$	Left down, right up
Odd	$a_n < 0$	Left up, right down

These four cases are sufficient to determine the behavior of every polynomial graph at infinity.

Examples.

Example 1. Determine the end behavior of

$$P(x) = 2x^5 - 3x^2 + 7.$$

Since the leading term is

$$2x^5,$$

the degree is odd and the leading coefficient is positive.
Therefore,

$$\boxed{\begin{aligned} x \to -\infty &\Longrightarrow P(x) \to -\infty, \\ x \to \infty &\Longrightarrow P(x) \to \infty. \end{aligned}}$$

Example 2. Determine the end behavior of

$$P(x) = -5x^6 + 3x - 1.$$

The leading term is

$$-5x^6.$$

The degree is even and the leading coefficient is negative.
Hence,

$$\boxed{\begin{aligned} x \to -\infty &\Longrightarrow P(x) \to -\infty, \\ x \to \infty &\Longrightarrow P(x) \to -\infty. \end{aligned}}$$

Both ends of the graph point downward.

6. Graphing Polynomial Functions

In the previous sections we learned how to determine the zeros of a polynomial, identify its degree, and predict its end behavior.
We now combine these ideas to sketch the graph of a polynomial function.
Unlike linear or quadratic functions, higher-degree polynomials can have many different shapes. Never-

theless, every polynomial graph can be understood by analyzing only a few important characteristics. These characteristics include

- the domain,
- the intercepts,
- the multiplicity of each zero,
- the end behavior,
- the maximum number of turning points.

Together these properties determine the overall appearance of the graph.

The Domain of a Polynomial. Every polynomial function is defined for every real number. Therefore,

$$\boxed{\text{Domain} = (-\infty, \infty).}$$

Unlike rational functions, there are no denominators that can become zero, and unlike radical functions, there are no restrictions imposed by square roots.

Finding the Intercepts. The first step in graphing any polynomial is locating its intercepts.

***y*-intercept.** Set

$$x = 0.$$

The point

$$(0, f(0))$$

is the y-intercept.

***x*-intercepts.** Set

$$f(x) = 0$$

and solve the equation.
Whenever possible, factor the polynomial completely.
Each real zero corresponds to an x-intercept.

6.1 Multiplicity of a Zero

Suppose

$$P(x) = (x - c)^m Q(x),$$

where

$$Q(c) \neq 0.$$

Then

$$x = c$$

is called a zero of multiplicity

$$m.$$

The multiplicity determines how the graph behaves near the intercept.

Odd Multiplicity. If the multiplicity is odd,

$$m = 1, 3, 5, \ldots,$$

the graph passes through the x-axis.
Examples include

$$(x-2), \qquad (x-2)^3, \qquad (x-2)^5.$$

The graph changes sign as it crosses the axis.

Even Multiplicity. If the multiplicity is even,

$$m = 2, 4, 6, \ldots,$$

the graph touches the x-axis and turns around.
Examples include

$$(x+1)^2, \qquad (x+1)^4.$$

The graph does not change sign.

Crossing Versus Touching. The following rule is extremely useful when sketching polynomial graphs.

Multiplicity	**Behavior**
1	Crosses the x-axis
2	Touches and turns
3	Crosses with flattening
4	Touches more gently

Higher multiplicities make the graph flatter near the zero.

Maximum Number of Turning Points. A turning point is a point where the graph changes direction. A polynomial of degree

$$n$$

can have at most

$$\boxed{n-1}$$

turning points.
For example,

Degree	Maximum Turning Points
2	1
3	2
4	3
5	4

This result helps determine whether a sketch is reasonable.

Example 1. Sketch the graph of

$$f(x) = (x+2)(x-1)^2.$$

Step 1. Determine the Degree. The degree is

$$1 + 2 = 3.$$

Therefore,

$$\boxed{\deg(f) = 3.}$$

Step 2. Determine the End Behavior. The leading term is

$$x^3.$$

Since the degree is odd and the leading coefficient is positive,

$$\boxed{\begin{array}{c} x \to -\infty \Longrightarrow f(x) \to -\infty, \\ x \to \infty \Longrightarrow f(x) \to \infty. \end{array}}$$

Step 3. Find the Zeros. The polynomial is already factored.
The zeros are

$$x = -2$$

and

$$x = 1.$$

The zero

$$-2$$

has multiplicity

$$1,$$

while

$$1$$

has multiplicity

$$2.$$

Step 4. Determine the Behavior at Each Zero. Since

$$x = -2$$

has odd multiplicity,
the graph crosses the x-axis.
Since

$$x = 1$$

has even multiplicity,
the graph touches the axis and turns around.

Step 5. Find the y-Intercept. Evaluate

$$f(0).$$

$$f(0) = (2)(1) = 2.$$

Hence,

$$\boxed{(0, 2)}$$

is the y-intercept.

Step 6. Sketch the Graph. The graph

- falls to the left,
- crosses at

$$x = -2,$$

- passes through

$$(0, 2),$$

- touches the axis at

$$x = 1,$$

- rises to the right.

These features uniquely determine the overall shape.

General Procedure for Graphing. The following checklist should be used whenever sketching a polynomial.

1. Determine the degree.
2. Determine the leading coefficient.
3. Determine the end behavior.
4. Find all real zeros.
5. Determine the multiplicity of each zero.
6. Decide whether the graph crosses or touches the axis.
7. Find the y-intercept.
8. Sketch the graph consistent with all of the above information.

7. Rational Functions

In the previous sections, we studied polynomial functions and learned how to analyze and graph them. We now turn our attention to a broader class of functions known as **rational functions**.

Rational functions arise naturally in mathematics, physics, engineering, economics, and many other fields. Unlike polynomial functions, rational functions may contain discontinuities, vertical asymptotes, horizontal asymptotes, slant asymptotes, and removable discontinuities (holes).

Understanding these features allows us to sketch accurate graphs and analyze their behavior.

7.1 Definition of a Rational Function

A rational function is any function that can be written as the quotient of two polynomials.

Definition 7.1. A function is called a **rational function** if it can be expressed in the form

$$\boxed{R(x) = \frac{P(x)}{Q(x)},}$$

where

- $P(x)$ and $Q(x)$ are polynomials,
- $Q(x) \neq 0$.

Notice that every polynomial function is technically a rational function, since

$$P(x) = \frac{P(x)}{1}.$$

However, when studying rational functions, we are usually interested in cases where the denominator contains the variable.

Examples of Rational Functions. Some common examples are

$$\frac{x+1}{x-2},$$

$$\frac{x^2+3}{x^2-4},$$

$$\frac{2x^3-x+5}{x^2+1},$$

and

$$\frac{x^2-9}{x^2-6x+9}.$$

Each of these functions is defined as the ratio of two polynomials.

Examples That Are Not Rational Functions. The following functions are **not** rational functions.

$$\sqrt{x},$$

because it contains a radical.

$$2^x,$$

because the variable appears in the exponent.

$$\sin x,$$

because it is a trigonometric function.

$$\ln x,$$

because it is a logarithmic function.

7.2 Domain of a Rational Function

The domain of a rational function consists of every real number for which the denominator is not zero. Therefore,

$$\boxed{\text{Domain} = \{\, x \in \mathbb{R} : Q(x) \neq 0 \,\}.}$$

To determine the domain, we simply solve

$$Q(x) = 0$$

and exclude those values.

Example 1. Find the domain of

$$f(x) = \frac{x+1}{x-3}.$$

Solution. The denominator is

$$x - 3.$$

Setting it equal to zero gives

$$x - 3 = 0,$$

so

$$x = 3.$$

Therefore,

$$x = 3$$

must be excluded.
Hence,

$$\boxed{(-\infty, 3) \cup (3, \infty).}$$

Example 2. Find the domain of

$$f(x) = \frac{x^2+4}{x^2-9}.$$

Solution. Factor the denominator.

$$x^2 - 9 = (x-3)(x+3).$$

The denominator is zero when

$$x = 3$$

or

$$x = -3.$$

Therefore,

$$\boxed{(-\infty, -3) \cup (-3, 3) \cup (3, \infty).}$$

Why the Domain Is Important. The values excluded from the domain determine many important features of the graph.
Depending on whether a factor cancels with the numerator, an excluded value may produce

- a vertical asymptote,

- or a removable discontinuity (hole).

These ideas will be discussed in the following sections.

7.3 Intercepts of Rational Functions

Once the domain of a rational function has been determined, the next step in analyzing its graph is to locate its intercepts.

The intercepts identify where the graph crosses the coordinate axes and provide important reference points for sketching the graph.

There are two types of intercepts.

- The **x-intercepts**, where the graph crosses the x-axis.
- The **y-intercept**, where the graph crosses the y-axis.

x-Intercepts. An x-intercept occurs whenever

$$y = 0.$$

For a rational function

$$f(x) = \frac{P(x)}{Q(x)},$$

the function equals zero only when the numerator is zero while the denominator is nonzero.

Therefore,

$$\boxed{\begin{aligned} P(x) &= 0, \\ Q(x) &\neq 0. \end{aligned}}$$

It is important to verify that the denominator is not zero after solving the numerator equation.

If both the numerator and denominator are zero at the same value of x, the graph contains a removable discontinuity (hole) rather than an x-intercept.

y-Intercept. The y-intercept is obtained by evaluating the function at

$$x = 0,$$

provided that

$$0$$

belongs to the domain.

Thus,

$$\boxed{y\text{-intercept} = (0, f(0)).}$$

If the denominator is zero at

$$x = 0,$$

then the function has no y-intercept.

Example 1. Find the intercepts of

$$f(x) = \frac{x+1}{x-2}.$$

Step 1. Find the x-Intercept. Set the numerator equal to zero.

$$x + 1 = 0.$$

Thus,

$$x = -1.$$

Check the denominator.

$$-1 - 2 = -3 \neq 0.$$

Therefore,

$$\boxed{(-1, 0)}$$

is the x-intercept.

Step 2. Find the y-Intercept. Evaluate

$$f(0).$$

$$f(0) = \frac{0+1}{0-2} = -\frac{1}{2}.$$

Hence,

$$\boxed{(0, -\tfrac{1}{2})}$$

is the y-intercept.

Example 2. Find the intercepts of

$$f(x) = \frac{x^2 - 9}{x^2 - 6x + 9}.$$

Step 1. Factor. Factor both the numerator and denominator.

$$x^2 - 9 = (x-3)(x+3),$$

and

$$x^2 - 6x + 9 = (x-3)^2.$$

Therefore,

$$f(x) = \frac{(x-3)(x+3)}{(x-3)^2}.$$

Step 2. Determine the x-Intercepts. The numerator is zero when

$$x = 3$$

or

$$x = -3.$$

However,

$$x = 3$$

also makes the denominator zero.
Therefore,

$$x = 3$$

is **not** an x-intercept.
The only remaining zero is

$$x = -3.$$

Thus,

$$\boxed{(-3, 0)}$$

is the only x-intercept.

Step 3. Determine the y-Intercept. Evaluate

$$f(0).$$

$$f(0) = \frac{-9}{9} = -1.$$

Hence,

$$\boxed{(0, -1)}$$

is the y-intercept.

7.4 Vertical Asymptotes

One of the distinguishing features of rational functions is the possible presence of vertical asymptotes.
A vertical asymptote occurs whenever the function becomes arbitrarily large (in the positive or negative direction) as the input approaches a certain value.
Graphically, the curve approaches a vertical line but never reaches it.

Definition. The vertical line

$$\boxed{x = a}$$

is called a **vertical asymptote** of the graph of a function if

$$\lim_{x \to a} f(x) = \pm\infty.$$

More precisely, at least one of the following limits must be infinite:

$$\lim_{x \to a^-} f(x), \qquad \lim_{x \to a^+} f(x).$$

If either limit is

$$+\infty$$

or

$$-\infty,$$

then the graph has a vertical asymptote at

$$x = a.$$

How to Find Vertical Asymptotes. For a rational function

$$f(x) = \frac{P(x)}{Q(x)},$$

vertical asymptotes occur at the zeros of the denominator that do **not** cancel with the numerator. The procedure is straightforward.

1. Factor both the numerator and denominator completely.
2. Cancel any common factors.
3. Set the remaining denominator equal to zero.
4. Solve for x.

Each remaining solution gives a vertical asymptote.

Important Remark. If a common factor cancels,

$$\frac{(x-2)(x+1)}{(x-2)(x-5)} = \frac{x+1}{x-5},$$

then

$$x = 2$$

is **not** a vertical asymptote.
Instead,
the graph contains a removable discontinuity (hole), which will be studied in the next section.
Only factors that remain in the denominator after simplification produce vertical asymptotes.

Example 1. Determine the vertical asymptotes of

$$f(x) = \frac{x+1}{x^2-9}.$$

Step 1. Factor the Denominator.

$$x^2 - 9 = (x-3)(x+3).$$

The function becomes

$$f(x) = \frac{x+1}{(x-3)(x+3)}.$$

Step 2. Check for Common Factors. The numerator

$$x + 1$$

shares no factor with the denominator.
Therefore,
no factors cancel.

Step 3. Solve the Denominator. Set

$$(x-3)(x+3)=0.$$

Thus,

$$x=3$$

or

$$x=-3.$$

Therefore,

$$\boxed{x=-3 \quad \text{and} \quad x=3}$$

are vertical asymptotes.

Example 2. Determine the vertical asymptotes of

$$f(x)=\frac{x^2-9}{x^2-6x+9}.$$

Step 1. Factor.

$$x^2-9=(x-3)(x+3),$$

$$x^2-6x+9=(x-3)^2.$$

Hence,

$$f(x)=\frac{(x-3)(x+3)}{(x-3)^2}.$$

Cancel one common factor.

$$f(x)=\frac{x+3}{x-3}.$$

Step 2. Solve the Remaining Denominator. The remaining denominator is

$$x-3.$$

Setting it equal to zero gives

$$x=3.$$

Therefore,

$$\boxed{x=3}$$

is the only vertical asymptote.
Notice that only one copy of

$$x-3$$

remained after cancellation.

Behavior Near a Vertical Asymptote. Knowing the location of a vertical asymptote is only part of the story.
We also want to know whether the graph approaches

$$+\infty$$

or

$$-\infty$$

on each side of the asymptote.
This is determined by examining the sign of the function immediately to the left and right of the asymptote.

Example. Consider

$$f(x) = \frac{1}{x-2}.$$

As

$$x \to 2^-,$$

the denominator is a very small negative number.
Therefore,

$$\boxed{\lim_{x\to 2^-} \frac{1}{x-2} = -\infty.}$$

As

$$x \to 2^+,$$

the denominator is a very small positive number.
Hence,

$$\boxed{\lim_{x\to 2^+} \frac{1}{x-2} = +\infty.}$$

The graph approaches opposite infinities from opposite sides of the asymptote.

Multiplicity and Vertical Asymptotes. The multiplicity of the denominator factor influences the behavior of the graph.

Odd Multiplicity. Suppose

$$f(x) = \frac{1}{(x-a)^3}.$$

The graph changes sign as it passes through the asymptote.
One side approaches

$$+\infty,$$

while the other approaches

$$-\infty.$$

Even Multiplicity. Suppose

$$f(x) = \frac{1}{(x-a)^2}.$$

Since

$$(x-a)^2 > 0$$

on both sides of

$$a,$$

both one-sided limits have the same sign.
Consequently,
the graph approaches the same infinity from both sides.

7.5 Removable Discontinuities (Holes)

Not every value excluded from the domain produces a vertical asymptote.
Sometimes a common factor appears in both the numerator and denominator. After simplifying the rational function, this common factor disappears. Although the simplified function is well-defined at most points, the original function is still undefined where the cancelled factor equals zero.
This type of discontinuity is called a **removable discontinuity**, or more commonly, a **hole**.

Definition. A removable discontinuity occurs whenever a common factor in the numerator and denominator cancels during simplification.
Suppose

$$f(x) = \frac{(x-a)P(x)}{(x-a)Q(x)},$$

where

$$Q(a) \neq 0.$$

Cancelling the common factor gives

$$f(x) = \frac{P(x)}{Q(x)}, \qquad x \neq a.$$

Although the simplified expression is perfectly well defined, the original function is still undefined at

$$x = a.$$

Consequently, the graph contains a hole at

$$x = a.$$

Finding the Coordinates of a Hole. The location of a hole is determined in two steps.

1. Factor the numerator and denominator completely.
2. Cancel every common factor.
3. Solve the cancelled factor to obtain the x-coordinate.
4. Substitute this value into the simplified function to obtain the y-coordinate.

Thus,

$$\boxed{\text{Hole} = (a,\, g(a))\,,}$$

where

$$g(x)$$

is the simplified function.

Example 1. Determine whether the function

$$f(x) = \frac{x^2 - 1}{x - 1}$$

contains a hole.

Step 1. Factor. Factor the numerator.

$$x^2 - 1 = (x - 1)(x + 1).$$

Therefore,

$$f(x) = \frac{(x - 1)(x + 1)}{x - 1}.$$

Step 2. Simplify. Cancel the common factor.

$$f(x) = x + 1, \qquad x \neq 1.$$

Notice that

$$x = 1$$

is still excluded because it belonged to the original denominator.

Step 3. Find the Hole. The cancelled factor gives

$$x = 1.$$

Evaluate the simplified function.

$$y = 1 + 1 = 2.$$

Therefore,

$$\boxed{(1, 2)}$$

is the location of the hole.

Example 2. Determine all discontinuities of

$$f(x) = \frac{x^2 - 9}{x^2 - 6x + 9}.$$

Step 1. Factor. Factor both numerator and denominator.

$$x^2 - 9 = (x - 3)(x + 3),$$

$$x^2 - 6x + 9 = (x-3)^2.$$

Hence,

$$f(x) = \frac{(x-3)(x+3)}{(x-3)^2}.$$

Step 2. Cancel Common Factors. Cancel one copy of

$$x - 3.$$

The simplified function is

$$f(x) = \frac{x+3}{x-3}.$$

Step 3. Analyze the Result. One factor

$$x - 3$$

remains in the denominator.
Therefore,

$$\boxed{x = 3}$$

is a vertical asymptote.
Notice that no factor has disappeared completely.
Consequently,

$$\boxed{\text{there is no hole.}}$$

Example 3. Determine all discontinuities of

$$f(x) = \frac{x^2 - 4}{x^2 - 2x}.$$

Step 1. Factor.

$$x^2 - 4 = (x-2)(x+2),$$

$$x^2 - 2x = x(x-2).$$

Therefore,

$$f(x) = \frac{(x-2)(x+2)}{x(x-2)}.$$

Step 2. Simplify. Cancel the common factor.

$$f(x) = \frac{x+2}{x}, \qquad x \neq 2.$$

Step 3. Identify the Discontinuities. The cancelled factor produces a hole.
Thus,

$$x = 2$$

is the x-coordinate of the hole.
Evaluate the simplified function.

$$\frac{2+2}{2} = 2.$$

Hence,

$$\boxed{(2,2)}$$

is the hole.
Since

$$x$$

remains in the denominator,

$$\boxed{x = 0}$$

is a vertical asymptote.

Holes Versus Vertical Asymptotes. It is important to distinguish between these two types of discontinuities.

Hole	**Vertical Asymptote**
$Common factor cancels$	$Factor remains in denominator$
$Single missing point$	$Function approaches \pm\infty$
$Finite limit exists$	$Infinite limit$

7.6 Horizontal Asymptotes

While vertical asymptotes describe the behavior of a rational function near values excluded from its domain, **horizontal asymptotes** describe the behavior of the function as

$$x \to \pm\infty.$$

They tell us which horizontal line the graph approaches far to the left or far to the right.
Unlike vertical asymptotes, a graph may cross a horizontal asymptote one or more times.

Definition. A horizontal line

$$\boxed{y = L}$$

is called a **horizontal asymptote** if

$$\boxed{\lim_{x\to\infty} f(x) = L}$$

or

$$\boxed{\lim_{x\to-\infty} f(x) = L.}$$

In other words, as

$$x$$

becomes extremely large (positively or negatively), the graph approaches the constant value

$$L.$$

Why Horizontal Asymptotes Exist. Consider the rational function

$$f(x) = \frac{P(x)}{Q(x)}.$$

For very large values of

$$|x|,$$

the highest-degree terms dominate all lower-degree terms.
Suppose

$$P(x) = a_n x^n + \cdots ,$$

and

$$Q(x) = b_m x^m + \cdots .$$

Then for large

$$|x|,$$

we have

$$f(x) \approx \frac{a_n x^n}{b_m x^m} = \frac{a_n}{b_m} x^{n-m}.$$

Therefore, the relationship between

$$n$$

and

$$m$$

completely determines the horizontal asymptote.

Case 1: Degree of Numerator Less Than Degree of Denominator. Suppose

$$\deg(P) < \deg(Q).$$

Then

$$n < m,$$

so

$$x^{n-m} = \frac{1}{x^{m-n}}.$$

Since

$$\frac{1}{x^{m-n}} \longrightarrow 0$$

as

$$x \to \pm\infty,$$

we obtain

$$\boxed{\lim_{x\to\pm\infty} f(x) = 0.}$$

Hence,

$$\boxed{y = 0}$$

is the horizontal asymptote.

Example.

$$f(x) = \frac{x+2}{x^2+1}.$$

The numerator has degree

$$1,$$

while the denominator has degree

$$2.$$

Therefore,

$$\boxed{y = 0.}$$

Case 2: Degrees Are Equal. Suppose

$$\deg(P) = \deg(Q).$$

Then

$$n = m.$$

The powers of

$$x$$

cancel,
leaving

$$\boxed{\lim_{x\to\pm\infty} f(x) = \frac{a_n}{b_m}.}$$

Therefore,

$$\boxed{y = \frac{a_n}{b_m}}$$

is the horizontal asymptote.

Example. Determine the horizontal asymptote of

$$f(x) = \frac{3x^2 + 5x - 1}{2x^2 - 7}.$$

Both numerator and denominator have degree

$$2.$$

The leading coefficients are

$$3$$

and

$$2.$$

Therefore,

$$\boxed{y = \frac{3}{2}.}$$

Case 3: Degree of Numerator Greater Than Degree of Denominator. Suppose

$$\deg(P) > \deg(Q).$$

Then

$$n > m.$$

The factor

$$x^{n-m}$$

grows without bound as

$$|x|$$

becomes large.
Consequently,

$$f(x)$$

does not approach a finite constant.
Therefore,

$$\boxed{\text{There is no horizontal asymptote.}}$$

In this situation, the function may instead possess a **slant asymptote** or another polynomial asymptote, which we study in the next section.

Example.

$$f(x) = \frac{x^3 + 2}{x + 1}.$$

Since

$$3 > 1,$$

there is

$$\boxed{\text{no horizontal asymptote.}}$$

7.7 Summary of the Three Cases

The three rules are summarized below.

Degrees	**Horizontal Asymptote**
$\deg(P) < \deg(Q)$	$y = 0$
$\deg(P) = \deg(Q)$	$y = \dfrac{\text{leading coefficient of } P}{\text{leading coefficient of } Q}$
$\deg(P) > \deg(Q)$	None

Students should memorize this table, but more importantly, understand that it follows directly from comparing the highest-degree terms of the polynomials.

Can a Graph Cross a Horizontal Asymptote?. Unlike vertical asymptotes, a graph **may cross** a horizontal asymptote.
A horizontal asymptote only describes the behavior of the graph for very large values of

$$|x|.$$

It places no restriction on the behavior of the graph elsewhere.
For example,

$$f(x) = \frac{x}{x^2 + 1}$$

has horizontal asymptote

$$y = 0,$$

yet

$$f(0) = 0,$$

so the graph actually crosses its horizontal asymptote at the origin.

7.8 Slant (Oblique) Asymptotes

In the previous section, we saw that a rational function does not have a horizontal asymptote when the degree of the numerator is greater than the degree of the denominator.
In one important case, however, the graph still approaches a straight line.
This line is called a **slant asymptote** or **oblique asymptote**.

Definition. A line

$$\boxed{y = mx + b}$$

is called a **slant asymptote** of a rational function if

$$\boxed{\lim_{x \to \pm\infty} [f(x) - (mx + b)] = 0.}$$

In other words, the graph becomes arbitrarily close to the line as

$$|x|$$

becomes very large.
Unlike a horizontal asymptote, the slant asymptote is not constant. Instead, it has a nonzero slope.

When Does a Slant Asymptote Exist?. A slant asymptote exists only when

$$\boxed{\deg(P) = \deg(Q) + 1,}$$

where

$$f(x) = \frac{P(x)}{Q(x)}.$$

That is, the numerator has degree exactly one greater than the denominator.
If the difference in degrees is two or more, the asymptote is no longer a line but a higher-degree polynomial.

Why Polynomial Division Works. Suppose

$$f(x) = \frac{P(x)}{Q(x)}.$$

Using polynomial division,

$$P(x) = Q(x)S(x) + R(x),$$

where

$$\deg(R) < \deg(Q).$$

Dividing by

$$Q(x),$$

we obtain

$$\boxed{f(x) = S(x) + \frac{R(x)}{Q(x)}.}$$

If

$$\deg(P) = \deg(Q) + 1,$$

then the quotient

$$S(x)$$

is a linear polynomial,

$$S(x) = mx + b.$$

Furthermore,

$$\deg(R) < \deg(Q),$$

which implies

$$\frac{R(x)}{Q(x)} \longrightarrow 0$$

as

$$x \to \pm\infty.$$

Consequently,

$$\boxed{f(x) \approx mx + b}$$

for large values of

$$|x|.$$

Therefore,
the quotient obtained from polynomial division is precisely the slant asymptote.

Procedure for Finding a Slant Asymptote. Whenever

$$\deg(P) = \deg(Q) + 1,$$

follow these steps.

1. Perform polynomial long division.
2. Ignore the remainder.
3. The quotient is the equation of the slant asymptote.

Example 1. Find the slant asymptote of

$$f(x) = \frac{x^2 + 3x + 2}{x + 1}.$$

Step 1. Compare Degrees. The numerator has degree

$$2,$$

and the denominator has degree

$$1.$$

Since

$$2 = 1 + 1,$$

a slant asymptote exists.

Step 2. Divide. Divide

$$x^2 + 3x + 2$$

by

$$x + 1.$$

Long division gives

$$x^2 + 3x + 2 = (x + 1)(x + 2).$$

Hence,

$$\boxed{f(x) = x + 2.}$$

There is no remainder.
Therefore,

$$\boxed{y = x + 2}$$

is the slant asymptote.
In this particular example, the graph actually coincides with the line, except at

$$x = -1,$$

where the original function is undefined.

Example 2. Find the slant asymptote of

$$f(x) = \frac{x^2 + 2}{x - 1}.$$

Step 1. Compare Degrees. Again,

$$2 = 1 + 1,$$

so a slant asymptote exists.

Step 2. Perform Long Division. Divide

$$x^2 + 2$$

by

$$x - 1.$$

The quotient is

$$x + 1,$$

and the remainder is

$$3.$$

Thus,

$$\boxed{\frac{x^2 + 2}{x - 1} = x + 1 + \frac{3}{x - 1}.}$$

Step 3. Determine the Asymptote. As

$$x \to \pm\infty,$$

the fraction

$$\frac{3}{x-1}$$

approaches zero.

Therefore,

$$\boxed{y = x + 1}$$

is the slant asymptote.

Relationship Between Horizontal and Slant Asymptotes. A rational function cannot have both a horizontal asymptote and a slant asymptote.

The type of asymptote depends entirely on the degrees of the numerator and denominator.

Degree Comparison	**Asymptote**
$\deg(P) < \deg(Q)$	$y = 0$
$\deg(P) = \deg(Q)$	Horizontal
$\deg(P) = \deg(Q) + 1$	Slant
$\deg(P) \geq \deg(Q) + 2$	Higher-degree polynomial asymptote

8. Graphing Rational Functions

We now combine all of the concepts developed throughout this chapter into a systematic procedure for graphing rational functions.

Unlike polynomial functions, rational functions may contain discontinuities, holes, and several types of asymptotes. Consequently, constructing an accurate graph requires analyzing each of these features separately before drawing the curve.

Fortunately, every rational function can be analyzed using the same sequence of steps.

A General Graphing Procedure. Suppose

$$f(x) = \frac{P(x)}{Q(x)}.$$

The graph may be sketched using the following algorithm.

1. Factor the numerator and denominator completely.
2. Determine the domain.
3. Simplify the rational function.
4. Locate any holes.
5. Find the x- and y-intercepts.
6. Determine the vertical asymptotes.
7. Determine the horizontal or slant asymptote.
8. Analyze the behavior near each asymptote.
9. Plot several additional points if necessary.
10. Sketch a smooth curve consistent with all information obtained.

This procedure should be followed whenever graphing a rational function.

Example 1. Sketch the graph of

$$f(x) = \frac{x+1}{x-2}.$$

Step 1. Domain. The denominator is zero when

$$x = 2.$$

Therefore,

$$\boxed{\text{Domain} = (-\infty, 2) \cup (2, \infty).}$$

Step 2. Intercepts. The numerator equals zero when

$$x = -1.$$

Thus,

$$\boxed{(-1, 0)}$$

is the x-intercept.
For the y-intercept,

$$f(0) = -\frac{1}{2}.$$

Hence,

$$\boxed{(0, -\tfrac{1}{2}).}$$

Step 3. Vertical Asymptote. The denominator is zero at

$$x = 2.$$

Therefore,

$$\boxed{x = 2}$$

is a vertical asymptote.
Step 4. Horizontal Asymptote. Both numerator and denominator have degree one. Therefore,

$$\boxed{y = 1.}$$

Step 5. Sketch. The graph

- approaches

  $$x = 2$$

  from both sides,
- approaches

  $$y = 1$$

  as

  $$x \to \pm\infty,$$

- passes through

$$(-1, 0)$$

and

$$(0, -\tfrac{1}{2}).$$

These features uniquely determine the graph.

Example 2. Sketch

$$f(x) = \frac{x^2 - 4}{x(x-2)}.$$

Step 1. Factor.

$$x^2 - 4 = (x-2)(x+2).$$

Therefore,

$$f(x) = \frac{(x-2)(x+2)}{x(x-2)}.$$

Step 2. Simplify. Cancel the common factor.

$$f(x) = \frac{x+2}{x}, \qquad x \neq 2.$$

Step 3. Hole. The cancelled factor gives

$$x = 2.$$

Evaluate the simplified function.

$$\frac{2+2}{2} = 2.$$

Hence,

$$\boxed{(2, 2)}$$

is a hole.

Step 4. Vertical Asymptote. The denominator is zero when

$$x = 0.$$

Thus,

$$\boxed{x = 0}$$

is the vertical asymptote.

Step 5. Horizontal Asymptote. The degrees are equal.
Therefore,

$$\boxed{y = 1.}$$

Step 6. Intercepts. The numerator is zero when

$$x = -2.$$

Thus,

$$\boxed{(-2, 0)}$$

is the only x-intercept.
There is no y-intercept because

$$x = 0$$

is not in the domain.

Example 3. Sketch

$$f(x) = \frac{x^2 + 2}{x - 1}.$$

Step 1. Domain.

$$x \neq 1.$$

Step 2. Intercepts. Since

$$x^2 + 2 > 0$$

for every real number,
there are no x-intercepts.
The y-intercept is

$$f(0) = -2.$$

Hence,

$$\boxed{(0, -2).}$$

Step 3. Vertical Asymptote.

$$\boxed{x = 1}$$

Step 4. Slant Asymptote. Perform polynomial division.

$$\frac{x^2 + 2}{x - 1} = x + 1 + \frac{3}{x - 1}.$$

Therefore,

$$\boxed{y = x + 1.}$$

Step 5. Sketch. The graph approaches

$$x = 1$$

near the discontinuity and approaches the line

$$y = x + 1$$

for large positive and negative values of

$$x.$$

9. Polynomial and Rational Inequalities

In previous sections, we learned how to determine the zeros of polynomial functions, locate the asymptotes of rational functions, and sketch their graphs.

These graphical techniques provide a powerful method for solving inequalities involving polynomial and rational expressions.

Instead of looking for the exact value where two expressions are equal, we now determine the intervals where an expression is positive or negative.

Such inequalities arise naturally in optimization, economics, engineering, physics, and many other applied sciences.

Polynomial Inequalities. A polynomial inequality has one of the forms

$$P(x) > 0, \qquad P(x) \geq 0, \qquad P(x) < 0, \qquad P(x) \leq 0,$$

where

$$P(x)$$

is a polynomial.

Since polynomials are continuous everywhere, their signs can change only at their real zeros.

Consequently, solving a polynomial inequality consists of determining the intervals where the polynomial is positive or negative.

Procedure. To solve a polynomial inequality,

1. Rewrite the inequality so that one side equals zero.
2. Factor the polynomial completely.
3. Determine all real zeros.
4. Divide the real line into intervals using these zeros.
5. Choose one test point from each interval.
6. Determine the sign of the polynomial on each interval.
7. Select the intervals satisfying the inequality.

Example. Solve

$$x^2 - 5x + 6 > 0.$$

Factor the polynomial.

$$x^2 - 5x + 6 = (x - 2)(x - 3).$$

The critical numbers are

$$x = 2, \qquad x = 3.$$

These divide the real line into the intervals

$$(-\infty, 2), \qquad (2, 3), \qquad (3, \infty).$$

Testing one point in each interval gives

Interval	Sign
$(-\infty, 2)$	$+$
$(2, 3)$	$-$
$(3, \infty)$	$+$

Since the inequality is

$$x^2 - 5x + 6 > 0,$$

the solution is

$$\boxed{(-\infty, 2) \cup (3, \infty).}$$

Rational Inequalities. A rational inequality has the form

$$\frac{P(x)}{Q(x)} > 0, \qquad \frac{P(x)}{Q(x)} < 0,$$

where

$$Q(x) \neq 0.$$

Unlike polynomial inequalities, the sign of a rational function may change at both

- zeros of the numerator, and
- zeros of the denominator.

Therefore, both types of critical numbers must be included in the sign chart.

Procedure. To solve a rational inequality,

1. Factor both the numerator and denominator.
2. Determine all zeros of the numerator.
3. Determine all zeros of the denominator.
4. Mark every critical number on a sign chart.
5. Test the sign of each interval.
6. Exclude values where the denominator is zero.
7. Include numerator zeros only when the inequality contains $\leq$ or $\geq$.

Example. Solve

$$\frac{x+1}{x-2} > 0.$$

The critical numbers are

$$x = -1, \qquad x = 2.$$

Testing the intervals gives

Interval	Sign
$(-\infty, -1)$	$+$
$(-1, 2)$	$-$
$(2, \infty)$	$+$

Therefore,

$$\boxed{(-\infty, -1) \cup (2, \infty).}$$

Notice that

$$x = 2$$

is never included because the denominator is zero there.

Sign Charts and the Test Interval Method. The most reliable method for solving polynomial and rational inequalities is the **sign chart method**. This method determines where an expression is positive or negative by examining its sign on intervals of the real line.

The basic idea is simple. The sign of a polynomial or rational function can only change at certain special points called *critical numbers*. Once these critical numbers are identified, they divide the real line into intervals. The sign of the expression remains constant throughout each interval.

Critical Numbers. The definition of a critical number depends on the type of expression.

Polynomial Functions. For a polynomial,

$$P(x),$$

the critical numbers are simply the real zeros of the polynomial.
That is,

$$P(x) = 0.$$

Rational Functions. For a rational function,

$$R(x) = \frac{P(x)}{Q(x)},$$

there are two kinds of critical numbers.

1. The zeros of the numerator,

$$P(x) = 0.$$

2. The zeros of the denominator,

$$Q(x) = 0.$$

The numerator zeros are possible solution points, while the denominator zeros are excluded from the domain.

Constructing a Sign Chart. Once all critical numbers have been found, construct the sign chart using the following procedure.

1. Arrange the critical numbers from smallest to largest.
2. Divide the real line into intervals determined by these numbers.
3. Select one convenient test point from each interval.
4. Substitute the test point into each factor.
5. Determine whether each factor is positive or negative.
6. Multiply the signs to determine the sign of the entire expression.
7. Choose the intervals satisfying the given inequality.

Example 1. Solve

$$(x - 1)(x + 2) > 0.$$

Step 1. Find the Critical Numbers. Set each factor equal to zero.

$$x = 1, \qquad x = -2.$$

Step 2. Divide the Real Line. The critical numbers divide the real line into

$$(-\infty, -2), \qquad (-2, 1), \qquad (1, \infty).$$

Step 3. Test Each Interval. Choose one test point from each interval.

Interval	$x-1$	$x+2$	Product
$(-\infty, -2)$	$-$	$-$	$+$
$(-2, 1)$	$-$	$+$	$-$
$(1, \infty)$	$+$	$+$	$+$

Step 4. Determine the Solution. Since the inequality is

$$(x-1)(x+2) > 0,$$

the product must be positive.
Therefore,

$$\boxed{(-\infty, -2) \cup (1, \infty).}$$

Example 2. Solve

$$\frac{(x-2)(x+1)}{x-3} < 0.$$

Step 1. Critical Numbers. The numerator is zero at

$$x = -1, \qquad x = 2,$$

and the denominator is zero at

$$x = 3.$$

Step 2. Divide the Real Line. The intervals are

$$(-\infty, -1), \quad (-1, 2), \quad (2, 3), \quad (3, \infty).$$

Step 3. Sign Chart.

Interval	Sign of $R(x)$
$(-\infty, -1)$	$-$
$(-1, 2)$	$+$
$(2, 3)$	$-$
$(3, \infty)$	$+$

Step 4. Determine the Solution. The inequality requires the expression to be negative.
Therefore,

$$\boxed{(-\infty, -1) \cup (2, 3).}$$

Notice that

$$x = 3$$

is excluded because it makes the denominator equal to zero.

Endpoint Rules. The treatment of endpoints depends on the type of inequality.

Point	Included?
Numerator zero	Included only for $\leq$, $\geq$
Denominator zero	Never included

Using Multiplicity to Determine the Sign. In many cases, it is not necessary to test every interval individually. The **multiplicity** of each factor determines whether the sign of the expression changes as the graph passes through a zero or a vertical asymptote.
Recall that if a factor appears more than once, its exponent is called its **multiplicity**.
For example,

$$(x-2)^3$$

has multiplicity 3, while

$$(x+1)^2$$

has multiplicity 2.

Odd Multiplicity. If a factor has an **odd multiplicity**, then its sign changes as the variable passes through the corresponding critical number.
For example,

$$(x-3)$$

is negative when

$$x < 3,$$

and positive when

$$x > 3.$$

Likewise,

$$(x-3)^3$$

also changes sign because raising a number to an odd power preserves its sign.
Consequently,

$$\boxed{\text{Odd multiplicity} \implies \text{the sign changes.}}$$

Even Multiplicity. If a factor has an **even multiplicity**, the sign does not change when passing through the corresponding critical number.
For example,

$$(x-3)^2$$

is always nonnegative.

Although the factor becomes zero at

$$x = 3,$$

it remains positive on both sides of the zero.
Therefore,

$$\boxed{\text{Even multiplicity} \implies \text{the sign does not change.}}$$

Sign Changes for Rational Functions. The same idea applies to rational functions.
Each factor in the numerator or denominator contributes to the sign of the function.
Whenever a factor has

- odd multiplicity, the sign changes;
- even multiplicity, the sign remains the same.

Notice that denominator zeros are still excluded from the domain, even though they may have even multiplicity.

Example 3. Determine the sign of

$$f(x) = \frac{(x-1)^2(x+2)}{(x-3)^3}.$$

without choosing test points.

Step 1. Identify the Critical Numbers. The critical numbers are

$$x = -2, \qquad x = 1, \qquad x = 3.$$

The multiplicities are

Factor	Multiplicity
$x+2$	1
$(x-1)^2$	2
$(x-3)^3$	3

Step 2. Determine Where the Sign Changes. Moving from left to right,

- at

  $$x = -2,$$

  the sign changes because the multiplicity is odd;
- at

  $$x = 1,$$

  the sign remains the same because the multiplicity is even;
- at

  $$x = 3,$$

  the sign changes again because the multiplicity is odd.

Assume the expression is positive for sufficiently large values of

$$x.$$

Working backward gives

Interval	Sign
$(-\infty, -2)$	$+$
$(-2, 1)$	$-$
$(1, 3)$	$-$
$(3, \infty)$	$+$

No test points were needed.

Advantages of the Multiplicity Method. When the factors and their multiplicities are known, the sign chart may be constructed much more quickly.

Instead of substituting test points into every interval, simply determine

1. whether each critical number has odd or even multiplicity,
2. whether the sign changes or remains the same,
3. propagate the signs across the intervals.

This method is especially useful when solving inequalities involving high-degree polynomials or rational functions with many factors.

10. Comprehensive Example

In this final example, we combine all of the techniques developed in this chapter to analyze a rational function completely.

Consider

$$f(x) = \frac{(x-2)(x+1)}{(x-1)(x+3)}.$$

Our goal is to determine the domain, intercepts, asymptotes, sign, end behavior, and sketch the graph.

Step 1. Domain. The denominator is

$$(x-1)(x+3).$$

Therefore,

$$x = 1$$

and

$$x = -3$$

must be excluded.

Hence,

$$\boxed{\text{Domain} = (-\infty, -3) \cup (-3, 1) \cup (1, \infty).}$$

Step 2. Intercepts. The numerator is zero when

$$x = 2$$

or

$$x = -1.$$

Since neither value makes the denominator zero, the x-intercepts are

$$\boxed{(-1,0), \qquad (2,0).}$$

The y-intercept is

$$f(0) = \frac{(-2)(1)}{(-1)(3)} = \frac{2}{3}.$$

Thus,

$$\boxed{(0, \tfrac{2}{3})}$$

is the y-intercept.

Step 3. Holes. There are no common factors between the numerator and denominator. Therefore,

$$\boxed{\text{There are no holes.}}$$

Step 4. Vertical Asymptotes. The denominator is zero when

$$x = -3$$

or

$$x = 1.$$

Since neither factor cancels,
both produce vertical asymptotes.

$$\boxed{x = -3, \qquad x = 1.}$$

Step 5. Horizontal Asymptote. The numerator and denominator both have degree two. Therefore,
the horizontal asymptote is the ratio of the leading coefficients.
Since both leading coefficients equal one,

$$\boxed{y = 1.}$$

Step 6. Sign Chart. The critical numbers are

$$-3, \qquad -1, \qquad 1, \qquad 2.$$

These divide the real line into five intervals.
Testing one point in each interval gives

Interval	Sign
$(-\infty, -3)$	$+$
$(-3, -1)$	$-$
$(-1, 1)$	$+$
$(1, 2)$	$-$
$(2, \infty)$	$+$

Step 7. Behavior Near the Asymptotes. Near

$$x = -3,$$

the graph approaches opposite infinities on opposite sides.
Likewise,
near

$$x = 1,$$

the graph also approaches opposite infinities.
As

$$x \to \pm\infty,$$

the graph approaches the horizontal asymptote

$$y = 1.$$

Step 8. Sketching the Graph. Using all of the information collected,
the graph

- has two disconnected branches separated by the vertical asymptotes,
- crosses the x-axis at

  $$(-1, 0)$$

  and

  $$(2, 0),$$

- crosses the y-axis at

  $$(0, \tfrac{2}{3}),$$

- approaches

  $$x = -3$$

  and

  $$x = 1$$

  without crossing them,
- approaches the horizontal asymptote

  $$y = 1$$

  as

  $$x \to \pm\infty.$$

A careful sketch should include all of these features.