Precalculus

Lecture 6 & 7: Trigonometric Functions

Angle Measure and Right-Triangle Trigonometry

Contents

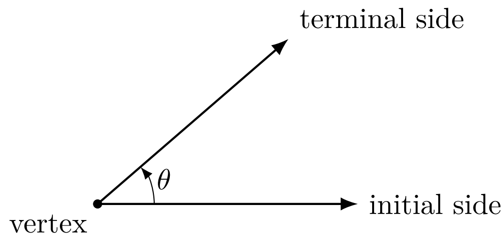
1. Angle Measure

Trigonometry begins with the measurement of angles. An angle may be viewed geometrically as two rays with a common endpoint, but it may also be viewed dynamically as a rotation from one ray to another. This second viewpoint is especially important because it allows angles to be positive, negative, and larger than one complete revolution.

Angles, Sides, and Vertices.

Definition 1.1. An **angle** is formed by two rays that share a common endpoint. The rays are called the **sides** of the angle, and their common endpoint is called the **vertex**.

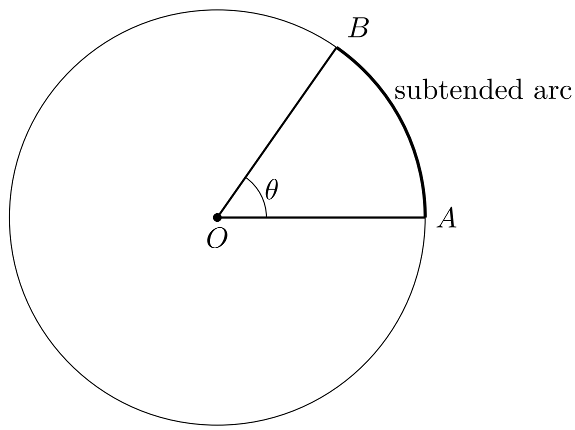
When the order of the two sides matters, one side is called the **initial side** and the other is called the **terminal side**. The angle measures the rotation required to move from the initial side to the terminal side.



Central Angles and Subtended Arcs.

Definition 1.2. A **central angle** of a circle is an angle whose vertex is at the center of the circle.

The portion of the circle cut off by the two sides of a central angle is called the **subtended arc**. Thus, a central angle determines an arc, and the length of that arc provides a natural way to measure the angle.



1.1 Degree Measure

Degree measure divides one complete revolution into 360 equal parts.

Definition 1.3. One degree, written $1°$, is the angle subtended by an arc whose length is $\frac{1}{360}$ of the circumference of a circle.

Consequently,

$$\boxed{1 \text{ complete revolution} = 360°.}$$

Several common angle classifications are listed below.

Type of Angle	Degree Measure
Acute	$0^\circ < \theta < 90^\circ$
Right	$\theta = 90^\circ$
Obtuse	$90^\circ < \theta < 180^\circ$
Straight	$\theta = 180^\circ$
Complete revolution	$\theta = 360^\circ$

1.2 Radian Measure

Degree measure is historically convenient, but radian measure is more natural in algebra, calculus, and applied mathematics because it compares arc length directly with radius.

Definition 1.4. The **radian measure** of a central angle is the ratio of the length s of the subtended arc to the radius r of the circle:

$$\boxed{\theta = \frac{s}{r}.}$$

If the arc length equals the radius, then

$$s = r,$$

and therefore

$$\theta = \frac{r}{r} = 1.$$

Thus, one radian is the central angle that subtends an arc whose length is equal to the radius of the circle.

Why $180^\circ = \pi$ Radians. A semicircle has arc length

$$s = \pi r.$$

Therefore,

$$\theta = \frac{s}{r} = \frac{\pi r}{r} = \pi.$$

Hence,

$$\boxed{180^\circ = \pi \text{ radians}, \qquad 360^\circ = 2\pi \text{ radians}.}$$

1.3 Converting Between Degrees and Radians

Since $180^\circ = \pi$, the conversion factors are

$$\boxed{\frac{\pi}{180^\circ} \qquad \text{and} \qquad \frac{180^\circ}{\pi}.}$$

Degrees to Radians. To convert degrees to radians, multiply by

$$\boxed{\frac{\pi}{180^\circ}.}$$

Example 1.. Convert 50° to radians.

Solution..

$$\begin{aligned}50^\circ &= 50^\circ\left(\frac{\pi}{180^\circ}\right)\\ &= \frac{50\pi}{180}\\ &= \boxed{\frac{5\pi}{18}}.\end{aligned}$$

Example 2.. Convert 225° to radians.

Solution..

$$\begin{aligned}225^\circ &= 225^\circ\left(\frac{\pi}{180^\circ}\right)\\ &= \frac{225\pi}{180}\\ &= \boxed{\frac{5\pi}{4}}.\end{aligned}$$

Radians to Degrees. To convert radians to degrees, multiply by

$$\boxed{\frac{180^\circ}{\pi}.}$$

Example 3.. Convert 3π radians to degrees.

Solution..

$$\begin{aligned}3\pi &= 3\pi\left(\frac{180^\circ}{\pi}\right)\\ &= \boxed{540^\circ}.\end{aligned}$$

Example 4.. Convert $-\frac{7\pi}{6}$ radians to degrees.

Solution..

$$\begin{aligned}-\frac{7\pi}{6} &= -\frac{7\pi}{6}\left(\frac{180^\circ}{\pi}\right)\\ &= \boxed{-210^\circ}.\end{aligned}$$

1.4 Oriented Angles

An oriented angle records both the amount and the direction of rotation.

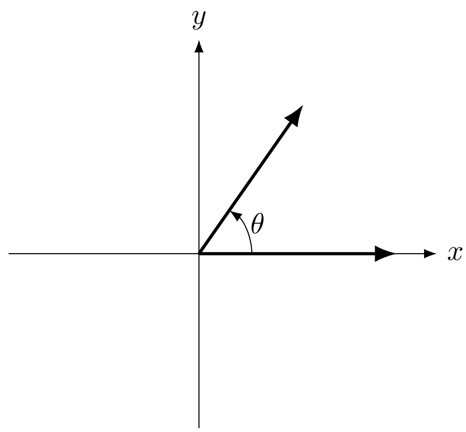
Definition 1.5. An angle is **positive** when the rotation from the initial side to the terminal side is counterclockwise. An angle is **negative** when the rotation is clockwise.

Direction of Rotation	Sign
Counterclockwise	Positive
Clockwise	Negative

For example, 30° and -330° end at the same terminal side. Similarly, $\frac{\pi}{4}$ and $-\frac{7\pi}{4}$ have the same terminal side.

1.5 Standard Position and Coterminal Angles

Definition 1.6. An angle is in **standard position** when its vertex is at the origin and its initial side lies on the positive x-axis.



Definition 1.7. Two angles in standard position are **coterminal** if they have the same terminal side.

All angles coterminal with θ are

$$\boxed{\theta + 360^\circ k, \qquad k \in \mathbb{Z},}$$

in degrees, or

$$\boxed{\theta + 2\pi k, \qquad k \in \mathbb{Z},}$$

in radians.

Example 5.. Show that 50°, 410°, and -310° are coterminal.

Solution..

$$410^\circ = 50^\circ + 360^\circ, \qquad -310^\circ = 50^\circ - 360^\circ.$$

Therefore, each differs from 50° by one complete revolution.

Example 6.. Find one positive and one negative angle coterminal with $\frac{\pi}{2}$.

Solution..

$$\frac{\pi}{2} + 2\pi = \boxed{\frac{5\pi}{2}}, \qquad \frac{\pi}{2} - 2\pi = \boxed{-\frac{3\pi}{2}}.$$

1.6 Arc Length

Theorem 1.8 (Arc-Length Formula)**.** *If a central angle of θ radians subtends an arc of length s in a circle of radius r, then*

$$\boxed{s = r\theta.}$$

The angle θ must be measured in radians.

Example 7: A Dog Walking Along a Circular Arc. A dog is attached to a pole by a leash 10 meters long. The dog walks along a circular arc subtending an angle of 70°. How far does the dog walk?

Solution..

$$70^\circ = 70^\circ \left(\frac{\pi}{180^\circ}\right) = \frac{7\pi}{18}.$$

Therefore,

$$\begin{aligned} s &= r\theta \\ &= 10\left(\frac{7\pi}{18}\right) \\ &= \boxed{\frac{35\pi}{9} \text{ m}} \\ &\approx 12.22 \text{ m.} \end{aligned}$$

Example 8: Finding the Central Angle. An arc of length 15 centimeters lies on a circle of radius 6 centimeters. Find the central angle in radians and degrees.

Solution..

$$\theta = \frac{s}{r} = \frac{15}{6} = \boxed{\frac{5}{2} \text{ radians}}.$$

In degrees,

$$\frac{5}{2}\left(\frac{180^\circ}{\pi}\right) = \frac{450^\circ}{\pi} \approx \boxed{143.24^\circ}.$$

Example 9: Finding the Radius. A central angle of $\frac{3\pi}{5}$ radians subtends an arc of length 12 inches. Find the radius.

Solution..

$$r = \frac{s}{\theta} = \frac{12}{3\pi/5} = \boxed{\frac{20}{\pi} \text{ in}} \approx 6.37 \text{ in.}$$

2. Trigonometric Functions

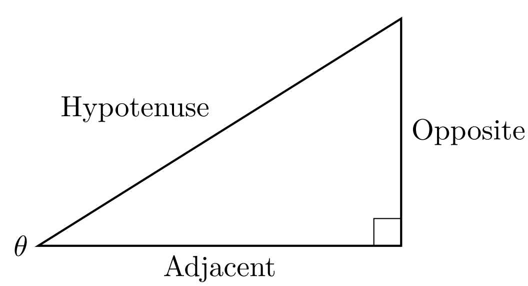
In the previous section we studied how angles are measured. We now introduce the trigonometric functions, which relate the angles of a right triangle to the ratios of its side lengths.

Trigonometric functions arise naturally in geometry, physics, engineering, computer graphics, navigation, and many other scientific disciplines. Their importance stems from the fact that they provide a connection between geometry and algebra.

2.1 Right Triangles

A **right triangle** is a triangle containing one right angle (90°). Relative to one of the acute angles, the three sides are named as follows.

- The **hypotenuse** is the side opposite the right angle.
- The **opposite side** is opposite the chosen acute angle.
- The **adjacent side** is next to the chosen acute angle.



2.2 Why Trigonometric Ratios Are Constant

One remarkable fact is that the ratios of the corresponding sides of similar right triangles are always equal.

Therefore, if two right triangles have the same acute angle θ, then

$$\frac{\text{opposite}}{\text{hypotenuse}}, \qquad \frac{\text{adjacent}}{\text{hypotenuse}}, \qquad \frac{\text{opposite}}{\text{adjacent}}$$

have exactly the same values regardless of the size of the triangle.
This observation motivates the definitions of the trigonometric functions.

2.3 The Six Trigonometric Functions

Let θ be an acute angle of a right triangle.

$$\boxed{\begin{aligned} \sin\theta &= \frac{\text{opposite}}{\text{hypotenuse}}, \\ \cos\theta &= \frac{\text{adjacent}}{\text{hypotenuse}}, \\ \tan\theta &= \frac{\text{opposite}}{\text{adjacent}}, \\ \csc\theta &= \frac{\text{hypotenuse}}{\text{opposite}}, \\ \sec\theta &= \frac{\text{hypotenuse}}{\text{adjacent}}, \\ \cot\theta &= \frac{\text{adjacent}}{\text{opposite}}. \end{aligned}}$$

A convenient memory aid is

$$\boxed{\text{SOH–CAH–TOA}}$$

where

- SOH: $\sin = \dfrac{O}{H}$,
- CAH: $\cos = \dfrac{A}{H}$,
- TOA: $\tan = \dfrac{O}{A}$.

2.4 Reciprocal Relationships

The remaining three trigonometric functions are reciprocals of the first three.

$$\boxed{\begin{aligned} \csc\theta &= \frac{1}{\sin\theta}, \\ \sec\theta &= \frac{1}{\cos\theta}, \\ \cot\theta &= \frac{1}{\tan\theta}. \end{aligned}}$$

2.5 Fundamental Identities

Using the definitions above,

$$\tan\theta = \frac{\sin\theta}{\cos\theta}, \qquad \cot\theta = \frac{\cos\theta}{\sin\theta}.$$

These identities are used extensively throughout trigonometry and calculus.

Example 1. A right triangle has side lengths 3, 4, and 5. Let θ be the angle opposite the side of length 3.
Find all six trigonometric functions.

Solution. The hypotenuse is 5, the opposite side is 3, and the adjacent side is 4.

$$\boxed{\begin{aligned} \sin\theta &= \frac{3}{5}, \\ \cos\theta &= \frac{4}{5}, \\ \tan\theta &= \frac{3}{4}, \\ \csc\theta &= \frac{5}{3}, \\ \sec\theta &= \frac{5}{4}, \\ \cot\theta &= \frac{4}{3}. \end{aligned}}$$

Example 2. Suppose

$$\sin\theta = \frac{5}{13},$$

where θ is an acute angle.
Find the remaining five trigonometric functions.

Solution. Since

$$\sin\theta = \frac{\text{opposite}}{\text{hypotenuse}} = \frac{5}{13},$$

take the opposite side to be 5 and the hypotenuse to be 13.
Using the Pythagorean Theorem,

$$a^2 + b^2 = c^2,$$

the adjacent side is

$$\sqrt{13^2 - 5^2} = \sqrt{169 - 25} = 12.$$

Therefore,

$$\boxed{\begin{aligned} \cos\theta &= \frac{12}{13}, \\ \tan\theta &= \frac{5}{12}, \\ \csc\theta &= \frac{13}{5}, \\ \sec\theta &= \frac{13}{12}, \\ \cot\theta &= \frac{12}{5}. \end{aligned}}$$

Example 3. A ladder 15 ft long leans against a wall. Its base is 9 ft from the wall.
Find the angle that the ladder makes with the ground.

Solution. The adjacent side is 9 ft and the hypotenuse is 15 ft.
Hence,

$$\cos\theta = \frac{9}{15} = \frac{3}{5}.$$

Therefore,

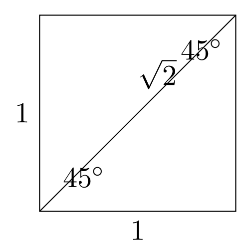
$$\boxed{\theta = \cos^{-1}\left(\frac{3}{5}\right) \approx 53.13^\circ.}$$

3. Special Right Triangles and Applications

Many trigonometric problems can be solved without a calculator by using two special right triangles whose side lengths follow simple geometric patterns.

3.1 The 45°–45°–90° Triangle

Consider a square of side length 1. Drawing a diagonal divides the square into two congruent right triangles.



By the Pythagorean Theorem,

$$1^2 + 1^2 = c^2$$

so that

$$c = \sqrt{2}.$$

Hence the side ratio is

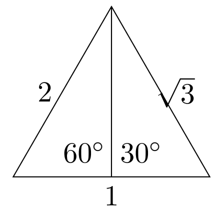
$$\boxed{1 : 1 : \sqrt{2}.}$$

Therefore,

$$\boxed{\begin{aligned} \sin 45^\circ &= \frac{\sqrt{2}}{2}, \\ \cos 45^\circ &= \frac{\sqrt{2}}{2}, \\ \tan 45^\circ &= 1. \end{aligned}}$$

3.2 The 30°–60°–90° Triangle

Start with an equilateral triangle of side length 2. Dropping an altitude creates two congruent right triangles.



Using the Pythagorean Theorem,

$$2^2 - 1^2 = 3,$$

so the altitude is

$$\sqrt{3}.$$

Thus the side ratio is

$$\boxed{1 : \sqrt{3} : 2.}$$

For the 30° angle,

$$\boxed{\begin{aligned} \sin 30^\circ &= \frac{1}{2}, \\ \cos 30^\circ &= \frac{\sqrt{3}}{2}, \\ \tan 30^\circ &= \frac{1}{\sqrt{3}} = \frac{\sqrt{3}}{3}. \end{aligned}}$$

For the 60° angle,

$$\boxed{\begin{aligned} \sin 60^\circ &= \frac{\sqrt{3}}{2}, \\ \cos 60^\circ &= \frac{1}{2}, \\ \tan 60^\circ &= \sqrt{3}. \end{aligned}}$$

3.3 Exact Values Table

θ	0°	30°	45°
$\sin\theta$	0	$\frac{1}{2}$	$\frac{\sqrt{2}}{2}$
$\cos\theta$	1	$\frac{\sqrt{3}}{2}$	$\frac{\sqrt{2}}{2}$
$\tan\theta$	0	$\frac{\sqrt{3}}{3}$	1

θ	60°	90°
$\sin\theta$	$\frac{\sqrt{3}}{2}$	1
$\cos\theta$	$\frac{1}{2}$	0
$\tan\theta$	$\sqrt{3}$	undefined

3.4 Solving Right Triangles

A right triangle is completely determined once two independent pieces of information are known, provided one of them is a side length.

The general strategy is

1. Sketch the triangle.
2. Label the known information.
3. Select the trigonometric function relating the known quantities to the unknown.
4. Solve the resulting equation.
5. Check that the answer is reasonable.

Example 1. A right triangle has hypotenuse 20 and an acute angle of 35°.
Find the remaining sides.

Solution. The opposite side satisfies

$$\sin 35^\circ = \frac{x}{20},$$

so

$$x = 20 \sin 35^\circ \approx 11.47.$$

The adjacent side satisfies

$$\cos 35^\circ = \frac{y}{20},$$

so

$$y = 20 \cos 35^\circ \approx 16.38.$$

Example 2. A tree casts a shadow 18 m long. The angle of elevation of the Sun is 40°.
Find the height of the tree.

Solution. Since

$$\tan 40^\circ = \frac{h}{18},$$

we obtain

$$h = 18 \tan 40^\circ \approx 15.10 \text{ m}.$$

Example 3. Standing 75 meters from a building, a student measures the angle of elevation to the top as 52°.
Find the height of the building.

Solution. Using tangent,

$$\tan 52^\circ = \frac{h}{75},$$

which gives

$$h = 75 \tan 52^\circ \approx 95.99 \text{ m}.$$

Key Formulas.

$$\sin\theta = \frac{O}{H}, \qquad \cos\theta = \frac{A}{H}, \qquad \tan\theta = \frac{O}{A},$$
$$\csc\theta = \frac{1}{\sin\theta}, \qquad \sec\theta = \frac{1}{\cos\theta}, \qquad \cot\theta = \frac{1}{\tan\theta},$$
$$\tan\theta = \frac{\sin\theta}{\cos\theta}.$$

4. Applications and Review

This section reinforces the ideas developed throughout the lecture by solving representative application problems. In each problem, the most important step is identifying which trigonometric ratio relates the known quantities to the unknown quantity.

4.1 Angles of Elevation and Depression

Definition 4.1. The **angle of elevation** is the angle measured upward from a horizontal line of sight. The **angle of depression** is the angle measured downward from a horizontal line of sight.

Since horizontal lines are parallel, angles of elevation and depression are often related by alternate interior angles.

4.2 Example: Airplane

An airplane is flying at an altitude of 2500 m. The angle of depression to a point on the ground is 18°. Find the horizontal distance from the airplane to the point.

Solution. Let x denote the horizontal distance.

$$\tan 18^\circ = \frac{2500}{x}.$$

Hence,

$$x = \frac{2500}{\tan 18^\circ} \approx 7694.2 \text{ m}.$$

4.3 Example: Guy Wire

A guy wire is attached to the top of a pole that is 18 m tall. If the wire makes an angle of 62° with the ground, determine the length of the wire.

Solution. Since the wire is the hypotenuse,

$$\sin 62^\circ = \frac{18}{L}.$$

Therefore,

$$L = \frac{18}{\sin 62^\circ} \approx 20.38 \text{ m}.$$

4.4 Calculator Notes

Before evaluating trigonometric functions, verify that your calculator is in the correct mode.

Problem	Calculator Mode
45°	Degree
120°	Degree
$\pi/6$	Radian
$5\pi/4$	Radian

4.5 Common Mistakes

1. Confusing the opposite and adjacent sides.
2. Using the wrong calculator mode.
3. Forgetting to convert degrees to radians before using $s = r\theta$.
4. Using tangent when sine or cosine is required.
5. Rounding too early during a computation.

4.6 Practice Problems

Solve each problem without looking at the solutions.

1. Convert 315° to radians.
2. Convert $-\dfrac{11\pi}{6}$ to degrees.
3. Find one positive and one negative coterminal angle for 145°.
4. Evaluate all six trigonometric functions for a 5-12-13 triangle.
5. A ladder 24 ft long rests against a wall with its base 7 ft from the wall. Find the angle the ladder makes with the ground.
6. Find the arc length subtended by a central angle of $\dfrac{7\pi}{12}$ in a circle of radius 9.
7. Find the height of a building if the angle of elevation is 41° from a point 60 m away.
8. Find the missing side of a right triangle if the hypotenuse is 18 and one acute angle is 27°.

4.7 Selected Answers

1. $\dfrac{7\pi}{4}$
2. $-330°$
3. $505°, -215°$
4. $\sin = \dfrac{5}{13}, \quad \cos = \dfrac{12}{13}, \quad \tan = \dfrac{5}{12}$
5. 73.04°
6. $\dfrac{21\pi}{4}$
7. 52.15 m
8. 16.04

4.8 Why Are Trigonometric Ratios Constant?

At first glance, it may appear surprising that quantities such as

$$\frac{\text{opposite}}{\text{hypotenuse}}$$

have a fixed value for a given angle. After all, there are infinitely many right triangles containing an acute angle of measure θ, and these triangles may have very different sizes.

The key observation is that all right triangles sharing the same acute angle are *similar*. Consequently, although the side lengths change, their corresponding ratios remain unchanged.

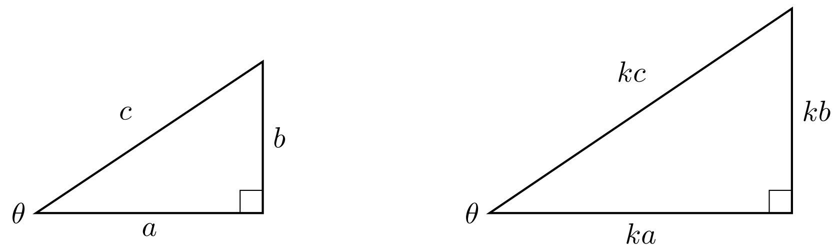
This important geometric fact is the foundation of trigonometry.

Theorem 4.2 (AA Similarity). *If two triangles have two corresponding angles equal, then the triangles are similar.*

Proof. The sum of the interior angles of every triangle is 180°. Therefore, if two angles of one triangle agree with two angles of another, the remaining angles must also be equal.

Since all three corresponding angles are equal, the triangles are similar. □

Suppose two right triangles each contain an acute angle θ.



Since the triangles are similar,

$$\frac{ka}{a} = \frac{kb}{b} = \frac{kc}{c} = k,$$

where k is the scale factor.
Now compare the ratio

$$\frac{\text{opposite}}{\text{hypotenuse}}.$$

For the first triangle,

$$\frac{b}{c}.$$

For the second triangle,

$$\frac{kb}{kc}.$$

Since the common factor k cancels,

$$\frac{kb}{kc} = \frac{b}{c}.$$

Therefore,

$$\boxed{\frac{\text{opposite}}{\text{hypotenuse}} \text{ depends only on the angle } \theta,}$$

and not on the size of the triangle.
Exactly the same reasoning shows that

$$\frac{\text{adjacent}}{\text{hypotenuse}}$$

and

$$\frac{\text{opposite}}{\text{adjacent}}$$

are also constant for a fixed angle.
This observation motivates the definitions of the six trigonometric functions.

Remark. The constancy of these ratios is one of the most important ideas in elementary mathematics. Without the concept of similar triangles, the trigonometric functions would not be well defined.

Example. Suppose two right triangles each contain a $40°$ angle.
The smaller triangle has

$$\text{opposite} = 3, \qquad \text{hypotenuse} = 5.$$

The larger triangle is obtained by doubling every side length.
Its opposite side is 6, and its hypotenuse is 10.
Then

$$\frac{3}{5} = 0.6,$$

while

$$\frac{6}{10} = 0.6.$$

Although the triangles are different sizes, the trigonometric ratio is exactly the same.
This illustrates why

$$\sin 40° = \frac{3}{5} = \frac{6}{10}.$$

The value depends only on the angle, not on the particular triangle.

4.9 The $45°$–$45°$–$90°$ Triangle

One of the most important right triangles in mathematics is the $45°$–$45°$–$90°$ **triangle**. This triangle is also called an *isosceles right triangle* because two of its sides have equal length.
The exact values of the trigonometric functions at $45°$ are obtained directly from this triangle and appear frequently throughout precalculus, calculus, engineering, and physics.

Review: Isosceles Triangles

Recall the following result from geometry.

Definition 4.3. An **isosceles triangle** is a triangle having at least two equal side lengths. The angles opposite those equal sides are also equal.

Suppose a right triangle has two equal legs.
Since one angle is already

$$90°,$$

the remaining two acute angles must add to

$$90°.$$

Because they are equal,

$$2\theta = 90°,$$

and therefore

$$\boxed{\theta = 45°.}$$

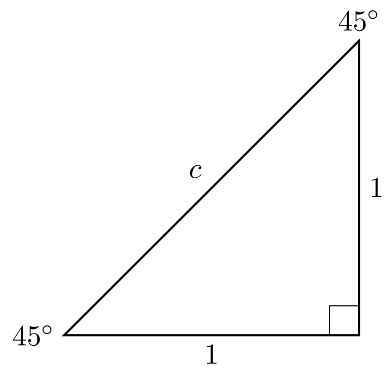
Hence every isosceles right triangle has angle measures

$$45^\circ, \qquad 45^\circ, \qquad 90^\circ.$$

Deriving the Side Lengths

Instead of memorizing the side lengths, let us derive them.
Assume each leg has length

$$1.$$



Using the Pythagorean Theorem,

$$a^2 + b^2 = c^2,$$

we obtain

$$1^2 + 1^2 = c^2,$$

which simplifies to

$$2 = c^2.$$

Taking the positive square root,

$$\boxed{c = \sqrt{2}.}$$

Therefore, every 45°–45°–90° triangle has side ratio

$$\boxed{1 : 1 : \sqrt{2}.}$$

Notice that this ratio remains true regardless of the size of the triangle because all such triangles are similar.

Deriving the Trigonometric Functions

Now let $\theta = 45^\circ$.
From the side lengths,

$$\text{opposite} = 1, \qquad \text{adjacent} = 1, \qquad \text{hypotenuse} = \sqrt{2}.$$

Therefore,

$$\sin 45^\circ = \frac{1}{\sqrt{2}}.$$

To simplify the denominator,

$$\frac{1}{\sqrt{2}} = \frac{1}{\sqrt{2}} \cdot \frac{\sqrt{2}}{\sqrt{2}} = \frac{\sqrt{2}}{2}.$$

Hence

$$\boxed{\sin 45^\circ = \frac{\sqrt{2}}{2}.}$$

Likewise,

$$\cos 45^\circ = \frac{1}{\sqrt{2}} = \boxed{\frac{\sqrt{2}}{2}}.$$

Finally,

$$\tan 45^\circ = \frac{1}{1} = \boxed{1}.$$

The reciprocal functions follow immediately.

$$\boxed{\begin{aligned} \csc 45^\circ &= \sqrt{2}, \\ \sec 45^\circ &= \sqrt{2}, \\ \cot 45^\circ &= 1. \end{aligned}}$$

Summary Table

Function	**Exact Value**
$\sin 45^\circ$	$\frac{\sqrt{2}}{2}$
$\cos 45^\circ$	$\frac{\sqrt{2}}{2}$
$\tan 45^\circ$	1
$\csc 45^\circ$	$\sqrt{2}$
$\sec 45^\circ$	$\sqrt{2}$
$\cot 45^\circ$	1

Example 1. Evaluate

$$\sin 45^\circ + \cos 45^\circ.$$

Solution. Using the exact values,

$$\sin 45^\circ + \cos 45^\circ = \frac{\sqrt{2}}{2} + \frac{\sqrt{2}}{2} = \boxed{\sqrt{2}.}$$

Example 2. Evaluate

$$\tan 45^\circ + \sec 45^\circ.$$

Solution. Since

$$\tan 45^\circ = 1,$$

and

$$\sec 45^\circ = \sqrt{2},$$

we obtain

$$\boxed{1 + \sqrt{2}.}$$

Example 3. Suppose a right triangle has one acute angle equal to 45° and hypotenuse 12. Find the remaining sides.

Solution. The side ratio is

$$1 : 1 : \sqrt{2}.$$

Therefore,

$$\frac{x}{12} = \frac{1}{\sqrt{2}},$$

which gives

$$x = \frac{12}{\sqrt{2}} = 6\sqrt{2}.$$

Hence both legs are

$$\boxed{6\sqrt{2}.}$$

Remark. You should memorize only one fact:

$$\boxed{1 : 1 : \sqrt{2}.}$$

All six trigonometric functions of 45° can be derived immediately from this ratio.

4.10 The 30°–60°–90° Triangle

The second special right triangle is the 30°–60°–90° **triangle**. Like the 45°–45°–90° triangle, it allows us to compute exact trigonometric values without using a calculator.
Rather than memorizing the side lengths, we shall derive them from an equilateral triangle.

Step 1: Begin with an Equilateral Triangle

Recall the following definition.

Definition 4.4. An **equilateral triangle** is a triangle whose three sides have the same length. Consequently, all three interior angles are equal and measure

$$60^\circ.$$

Assume each side has length

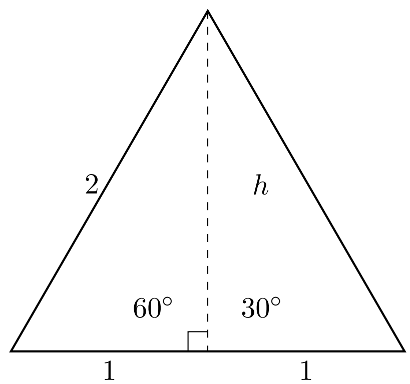
$$2.$$

The choice of 2 is intentional because it simplifies the calculations.

Step 2: Construct an Altitude

Draw an altitude from the top vertex.
The altitude

- bisects the opposite side,
- bisects the top angle,
- forms two congruent right triangles.



Notice that the altitude divides

$$60^\circ$$

into two equal angles,

$$30^\circ.$$

Thus each right triangle has angles

$$30^\circ, \qquad 60^\circ, \qquad 90^\circ.$$

Step 3: Compute the Altitude

Since the altitude forms a right triangle,
the legs are

$$1 \quad \text{and} \quad h,$$

while the hypotenuse is

$$2.$$

Applying the Pythagorean Theorem,

$$1^2 + h^2 = 2^2,$$

gives

$$1 + h^2 = 4.$$

Therefore,

$$h^2 = 3,$$

and

$$\boxed{h = \sqrt{3}.}$$

Hence every 30°–60°–90° triangle has side ratio

$$\boxed{1 : \sqrt{3} : 2.}$$

This is one of the most important ratios in all of trigonometry.

Exact Trigonometric Values for 30°

Let

$$\theta = 30^\circ.$$

From the triangle,

$$\text{opposite} = 1,$$

$$\text{adjacent} = \sqrt{3},$$

$$\text{hypotenuse} = 2.$$

Therefore,

$$\boxed{\begin{aligned} \sin 30^\circ &= \frac{1}{2}, \\ \cos 30^\circ &= \frac{\sqrt{3}}{2}, \\ \tan 30^\circ &= \frac{1}{\sqrt{3}} = \frac{\sqrt{3}}{3}. \end{aligned}}$$

Using reciprocal identities,

$$\boxed{\begin{aligned} \csc 30^\circ &= 2, \\ \sec 30^\circ &= \frac{2}{\sqrt{3}} = \frac{2\sqrt{3}}{3}, \\ \cot 30^\circ &= \sqrt{3}. \end{aligned}}$$

Exact Trigonometric Values for $60°$

Now let

$$\theta = 60°.$$

The opposite side is now

$$\sqrt{3},$$

while the adjacent side is

$$1.$$

Hence,

$$\boxed{\begin{aligned} \sin 60° &= \frac{\sqrt{3}}{2}, \\ \cos 60° &= \frac{1}{2}, \\ \tan 60° &= \sqrt{3}. \end{aligned}}$$

The reciprocal functions are

$$\boxed{\begin{aligned} \csc 60° &= \frac{2}{\sqrt{3}} = \frac{2\sqrt{3}}{3}, \\ \sec 60° &= 2, \\ \cot 60° &= \frac{1}{\sqrt{3}} = \frac{\sqrt{3}}{3}. \end{aligned}}$$

Summary Table

Function	$30°$	$60°$
sin	$\frac{1}{2}$	$\frac{\sqrt{3}}{2}$
cos	$\frac{\sqrt{3}}{2}$	$\frac{1}{2}$
tan	$\frac{\sqrt{3}}{3}$	$\sqrt{3}$
csc	2	$\frac{2\sqrt{3}}{3}$
sec	$\frac{2\sqrt{3}}{3}$	2
cot	$\sqrt{3}$	$\frac{\sqrt{3}}{3}$

Example 1. Evaluate

$$2\sin 30^\circ + \cos 60^\circ.$$

Solution. Since

$$\sin 30^\circ = \frac{1}{2}, \qquad \cos 60^\circ = \frac{1}{2},$$

we obtain

$$2\left(\frac{1}{2}\right) + \frac{1}{2} = 1 + \frac{1}{2} = \boxed{\frac{3}{2}.}$$

Example 2. Evaluate

$$\tan 60^\circ + \cot 30^\circ.$$

Solution. Both values equal

$$\sqrt{3}.$$

Hence,

$$\boxed{2\sqrt{3}.}$$

Example 3. A right triangle has hypotenuse

$$18$$

and one acute angle

$$30^\circ.$$

Find the remaining two sides.

Solution. Using the side ratio

$$1 : \sqrt{3} : 2,$$

we have

$$\frac{x}{18} = \frac{1}{2},$$

so

$$x = 9.$$

Similarly,

$$\frac{y}{18} = \frac{\sqrt{3}}{2},$$

giving

$$y = 9\sqrt{3}.$$

Therefore,

$$\boxed{\text{legs} = 9, \qquad 9\sqrt{3}.}$$

Remark. Rather than memorizing six separate values, remember only the side ratio

$$\boxed{1 : \sqrt{3} : 2.}$$

Every trigonometric function of 30° and 60° can be obtained immediately from this ratio.

5. Basic Trigonometric Identities

In mathematics, an identity is an equation that is true for every value of the variable for which both sides are defined.
Unlike an ordinary equation, which may be true only for certain values, an identity is always true.
For example,

$$x + 2 = 7$$

is an equation that is true only when

$$x = 5.$$

On the other hand,

$$(a + b)^2 = a^2 + 2ab + b^2$$

is an identity because it is true for every choice of a and b.
Trigonometric identities play a central role in algebra, calculus, differential equations, and engineering.

5.1 Reciprocal Identities

Recall the definitions of the six trigonometric functions.

$$\sin\theta = \frac{\text{opposite}}{\text{hypotenuse}},$$

and

$$\csc\theta = \frac{\text{hypotenuse}}{\text{opposite}}.$$

Since these are reciprocals,

$$\boxed{\csc\theta = \frac{1}{\sin\theta}.}$$

Similarly,

$$\boxed{\sec\theta = \frac{1}{\cos\theta}, \qquad \cot\theta = \frac{1}{\tan\theta}.}$$

These are called the **reciprocal identities**.

5.2 Quotient Identities

The quotient identities follow directly from the definitions of sine and cosine.
Observe that

$$\sin\theta = \frac{O}{H},$$

and

$$\cos\theta = \frac{A}{H}.$$

Therefore,

$$\frac{\sin\theta}{\cos\theta} = \frac{O/H}{A/H}.$$

Since the hypotenuse appears in both numerator and denominator, it cancels.
Hence,

$$\boxed{\tan\theta = \frac{\sin\theta}{\cos\theta}.}$$

Likewise,

$$\boxed{\cot\theta = \frac{\cos\theta}{\sin\theta}.}$$

5.3 Proof of the Pythagorean Identity

The most famous identity in trigonometry is

$$\boxed{\sin^2\theta + \cos^2\theta = 1.}$$

It is called the **Pythagorean Identity** because it comes directly from the Pythagorean Theorem.
Consider a right triangle.
Let

$$\text{hypotenuse} = c,$$

$$\text{opposite} = a,$$

$$\text{adjacent} = b.$$

Then

$$a^2 + b^2 = c^2.$$

Divide every term by

$$c^2.$$

This gives

$$\frac{a^2}{c^2} + \frac{b^2}{c^2} = 1.$$

Notice that

$$\frac{a}{c} = \sin\theta,$$

and

$$\frac{b}{c} = \cos\theta.$$

Therefore,

$$\boxed{\sin^2\theta + \cos^2\theta = 1.}$$

5.4 Deriving the Remaining Pythagorean Identities

Starting with

$$\sin^2\theta + \cos^2\theta = 1,$$

divide every term by

$$\cos^2\theta.$$

Then

$$\frac{\sin^2\theta}{\cos^2\theta} + 1 = \frac{1}{\cos^2\theta}.$$

Recognizing the trigonometric ratios,

$$\boxed{\tan^2\theta + 1 = \sec^2\theta.}$$

Similarly, divide

$$\sin^2\theta + \cos^2\theta = 1$$

by

$$\sin^2\theta.$$

Then

$$1 + \frac{\cos^2\theta}{\sin^2\theta} = \frac{1}{\sin^2\theta}.$$

Hence,

$$\boxed{1 + \cot^2\theta = \csc^2\theta.}$$

5.5 Complementary Identities

Two angles are said to be **complementary** if their measures add to $90°$.
In every right triangle,
the two acute angles are complementary.
Suppose one acute angle is

$$\theta.$$

Then the other is

$$90^\circ - \theta.$$

Notice what happens to the sides.
The side opposite one angle becomes the adjacent side of the other.
Therefore,

$$\boxed{\sin(90^\circ - \theta) = \cos\theta.}$$

Similarly,

$$\boxed{\cos(90^\circ - \theta) = \sin\theta.}$$

These identities are extremely useful throughout calculus.

5.6 Summary of Fundamental Identities

$$\boxed{\begin{aligned} \tan\theta &= \frac{\sin\theta}{\cos\theta}, \\ \cot\theta &= \frac{\cos\theta}{\sin\theta}, \\ \sec\theta &= \frac{1}{\cos\theta}, \\ \csc\theta &= \frac{1}{\sin\theta}, \\ \sin^2\theta + \cos^2\theta &= 1, \\ 1 + \tan^2\theta &= \sec^2\theta, \\ 1 + \cot^2\theta &= \csc^2\theta, \\ \sin(90^\circ - \theta) &= \cos\theta, \\ \cos(90^\circ - \theta) &= \sin\theta. \end{aligned}}$$

Example 1. If

$$\sin\theta = \frac{3}{5},$$

find

$$\csc\theta.$$

Solution. Using the reciprocal identity,

$$\boxed{\csc\theta = \frac{5}{3}.}$$

Example 2. Given

$$\sin\theta = \frac{4}{5},$$

find

$$\cos\theta.$$

Solution. Using the Pythagorean Identity,

$$\sin^2\theta + \cos^2\theta = 1,$$

we have

$$\left(\frac{4}{5}\right)^2 + \cos^2\theta = 1.$$

Thus,

$$\frac{16}{25} + \cos^2\theta = 1,$$

so

$$\cos^2\theta = \frac{9}{25}.$$

Since the angle is acute,

$$\boxed{\cos\theta = \frac{3}{5}.}$$

Example 3. Evaluate

$$\sin 30^\circ + \cos 60^\circ.$$

Solution. Using the exact values,

$$\frac{1}{2} + \frac{1}{2} = \boxed{1.}$$

Example 4. Simplify

$$\frac{\sin\theta}{\cos\theta}.$$

Solution. Using the quotient identity,

$$\boxed{\tan\theta.}$$

Remark. Students often try to memorize all nine identities separately.
This is unnecessary.
If you remember

$$\boxed{\sin^2\theta + \cos^2\theta = 1,}$$

together with the definitions of the six trigonometric functions, every other identity in this section can be derived in only a few steps.
Understanding is far more valuable than memorization.

6. Solving Right Triangles

One of the primary applications of trigonometry is determining unknown lengths and angles of a right triangle.

A right triangle is said to be **completely solved** when the measures of all three sides and all three angles are known.

Since one angle of every right triangle is 90°, only two acute angles remain unknown. Furthermore, these two angles are complementary, so knowing one automatically determines the other.

Consequently, solving a right triangle means finding

- the missing side lengths, and
- the missing acute angle.

6.1 General Strategy

When solving any right-triangle problem, follow the same sequence of steps.

1. Draw and label a diagram.
2. Identify the known quantities.
3. Determine which side is opposite, adjacent, and hypotenuse relative to the given angle.
4. Choose the trigonometric function involving only the known and unknown quantities.
5. Solve the resulting equation.
6. Check that the answer is reasonable.

Students often make mistakes because they choose a trigonometric function before carefully labeling the triangle.

6.2 Choosing the Correct Trigonometric Function

The following table summarizes when each trigonometric function should be used.

Known Quantities	**Use**
Opposite and Hypotenuse	$\sin\theta$
Adjacent and Hypotenuse	$\cos\theta$
Opposite and Adjacent	$\tan\theta$

If an unknown angle is required, use the corresponding inverse trigonometric function.

6.3 Example 1: Finding a Side

A right triangle has hypotenuse

$$18$$

and an acute angle of

$$40^\circ.$$

Find the length of the side opposite the angle.

Solution. Since the opposite side and the hypotenuse are involved, use sine.

$$\sin 40^\circ = \frac{x}{18}.$$

Multiplying both sides by 18,

$$x = 18\sin 40^\circ.$$

Using a calculator,

$$\boxed{x \approx 11.57.}$$

6.4 Example 2: Finding Another Side

Using the same triangle, find the adjacent side.

Solution. Now use cosine.

$$\cos 40^\circ = \frac{x}{18}.$$

Hence,

$$x = 18 \cos 40^\circ.$$

Therefore,

$$\boxed{x \approx 13.79.}$$

6.5 Example 3: Finding an Angle

A right triangle has

$$\text{adjacent} = 12, \qquad \text{hypotenuse} = 20.$$

Find the acute angle.

Solution. Use cosine.

$$\cos \theta = \frac{12}{20} = 0.6.$$

Apply the inverse cosine function.

$$\theta = \cos^{-1}(0.6).$$

Therefore,

$$\boxed{\theta \approx 53.13^\circ.}$$

The remaining acute angle is

$$90^\circ - 53.13^\circ = 36.87^\circ.$$

6.6 Example 4: Ladder Problem

A ladder is

$$24 \text{ ft}$$

long and rests against a vertical wall.
Its base is

$$7 \text{ ft}$$

from the wall.
Find the angle that the ladder makes with the ground.

Solution. The adjacent side and hypotenuse are known.

Therefore,

$$\cos\theta = \frac{7}{24}.$$

Applying the inverse cosine,

$$\theta = \cos^{-1}\left(\frac{7}{24}\right).$$

Hence,

$$\boxed{\theta \approx 73.04^\circ.}$$

6.7 Example 5: Height of a Building

Standing

$$80 \text{ meters}$$

from a building, a surveyor measures the angle of elevation to the top to be

$$38^\circ.$$

Find the height of the building.

Solution. The unknown height is opposite the angle.
The horizontal distance is adjacent.
Thus,

$$\tan 38^\circ = \frac{h}{80}.$$

Multiplying both sides by 80,

$$h = 80\tan 38^\circ.$$

Therefore,

$$\boxed{h \approx 62.50 \text{ m.}}$$

6.8 Example 6: Guy Wire

A guy wire is attached to the top of a pole that is

$$15 \text{ m}$$

high.
The wire makes an angle of

$$65^\circ$$

with the ground.
Find the length of the wire.

Solution. The wire is the hypotenuse.
Hence,

$$\sin 65^\circ = \frac{15}{L}.$$

Therefore,

$$L = \frac{15}{\sin 65^\circ}.$$

Thus,

$$\boxed{L \approx 16.55 \text{ m.}}$$