Precalculus

Lecture 8 & 9

§6.5–6.6 The Laws of Sines and Cosines,
§7.1–7.3 Trigonometric Identities and Angle Formulas,

§7.4–7.5 Trigonometric Equations

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1. The Law of Sines

Triangles arise naturally in surveying, navigation, engineering, physics, and many other applications. While the Pythagorean Theorem can be used to solve right triangles, it is not applicable to general (oblique) triangles. For non-right triangles, one of the most important relationships is the **Law of Sines**, which connects the sides of a triangle with their opposite angles.

Throughout this section, let a triangle have

$$\text{angles } \alpha,\ \beta,\ \gamma,$$

and opposite sides

$$a,\ b,\ c,$$

respectively.

1.1 The Law of Sines

Theorem 1.1 (The Law of Sines)**.** *For any triangle with sides a, b, and c, and opposite angles α, β, and γ, respectively,*

$$\boxed{\frac{\sin\alpha}{a} = \frac{\sin\beta}{b} = \frac{\sin\gamma}{c}}$$

or equivalently,

$$\boxed{\frac{a}{\sin\alpha} = \frac{b}{\sin\beta} = \frac{c}{\sin\gamma}.}$$

The second form is often more convenient when solving problems, although both equations express exactly the same relationship.

The Law of Sines tells us that the ratio of the length of a side to the sine of its opposite angle is constant throughout the triangle. Consequently, if one side-angle pair is known, the remaining sides or angles can often be determined.

1.2 When Can the Law of Sines Be Used?

The Law of Sines is particularly useful when the given information contains at least one known side and its opposite angle. Common situations include

- **ASA (Angle-Side-Angle):** two angles and the included side.
- **AAS (Angle-Angle-Side):** two angles and a non-included side.
- **SSA (Side-Side-Angle):** two sides and an angle opposite one of the sides. This case is known as the *ambiguous case* because it may produce zero, one, or two possible triangles.

We begin with the simpler AAS/ASA cases before discussing the ambiguous case.

1.3 The AAS (or ASA) Case

Suppose two angles and one side of a triangle are known. Since the sum of the interior angles of every triangle is

$$180^\circ,$$

the missing angle can always be found first:

$$\boxed{\gamma = 180^\circ - \alpha - \beta.}$$

Once all three angles are known, the Law of Sines may be used to determine each unknown side.

The general procedure is:

1. Compute the missing angle using the angle-sum property.
2. Select a known side-angle pair.
3. Apply the Law of Sines to determine one unknown side.
4. Repeat the process until the triangle has been completely solved.

1.4 Example 1

Example 1.2. Solve the triangle if

$$\alpha = 30^\circ, \qquad \beta = 50^\circ, \qquad a = 10.$$

Step 1: Find the third angle.
Since the angles of a triangle sum to 180°,

$$\gamma = 180^\circ - 30^\circ - 50^\circ = 100^\circ.$$

Step 2: Find side b.
Using the Law of Sines,

$$\frac{a}{\sin\alpha} = \frac{b}{\sin\beta},$$

we obtain

$$b = \frac{a\sin\beta}{\sin\alpha}.$$

Substituting the known values,

$$b = \frac{10\sin 50^\circ}{\sin 30^\circ} = 20\sin 50^\circ.$$

Using a calculator,

$$\boxed{b \approx 15.32.}$$

Step 3: Find side c.
Again applying the Law of Sines,

$$\frac{a}{\sin\alpha} = \frac{c}{\sin\gamma},$$

gives

$$c = \frac{10\sin 100^\circ}{\sin 30^\circ} = 20\sin 100^\circ.$$

Therefore,

$$\boxed{c \approx 19.70.}$$

Hence the completed triangle is

$$\boxed{\alpha = 30^\circ, \qquad \beta = 50^\circ, \qquad \gamma = 100^\circ,}$$

$$\boxed{a = 10, \qquad b \approx 15.32, \qquad c \approx 19.70.}$$

1.5 Example 2

Example 1.3. Solve the triangle if

$$\alpha = 70^\circ, \qquad \beta = 50^\circ, \qquad c = 13.$$

We are given two angles and one side, so this is an AAS case. We first determine the remaining angle and then use the Law of Sines to find the unknown sides.

Step 1: Find the third angle.

Because the interior angles of a triangle sum to 180°,

$$\gamma = 180^\circ - \alpha - \beta.$$

Substituting the known values,

$$\gamma = 180^\circ - 70^\circ - 50^\circ = 60^\circ.$$

Thus,

$$\boxed{\gamma = 60^\circ.}$$

Step 2: Find side a.

The side a is opposite the angle $\alpha = 70^\circ$, while the known side $c = 13$ is opposite the angle $\gamma = 60^\circ$. Using the Law of Sines,

$$\frac{a}{\sin\alpha} = \frac{c}{\sin\gamma}.$$

Solving for a gives

$$a = \frac{c\sin\alpha}{\sin\gamma}.$$

Substitute the known values:

$$a = \frac{13\sin 70^\circ}{\sin 60^\circ}.$$

Since

$$\sin 60^\circ = \frac{\sqrt{3}}{2},$$

we have

$$a = \frac{13\sin 70^\circ}{\sqrt{3}/2} = \frac{26}{\sqrt{3}}\sin 70^\circ.$$

Rationalizing the denominator,

$$a = \frac{26\sqrt{3}}{3}\sin 70^\circ.$$

Therefore,

$$\boxed{a = \frac{26\sqrt{3}}{3}\sin 70^\circ}$$

and numerically,

$$\boxed{a \approx 14.10.}$$

Step 3: Find side b.
The side b is opposite the angle $\beta = 50°$. Again using the Law of Sines,

$$\frac{b}{\sin\beta} = \frac{c}{\sin\gamma}.$$

Solving for b,

$$b = \frac{c\sin\beta}{\sin\gamma}.$$

Substitute the known values:

$$b = \frac{13\sin 50°}{\sin 60°}.$$

Using $\sin 60° = \sqrt{3}/2$,

$$b = \frac{13\sin 50°}{\sqrt{3}/2} = \frac{26}{\sqrt{3}}\sin 50°.$$

Rationalizing the denominator gives

$$b = \frac{26\sqrt{3}}{3}\sin 50°.$$

Therefore,

$$\boxed{b = \frac{26\sqrt{3}}{3}\sin 50°}$$

and numerically,

$$\boxed{b \approx 11.50.}$$

The completed triangle is

$$\boxed{\alpha = 70°, \qquad \beta = 50°, \qquad \gamma = 60°,}$$

with side lengths

$$\boxed{a \approx 14.10, \qquad b \approx 11.50, \qquad c = 13.}$$

1.6 The Ambiguous SSA Case

The Law of Sines can also be used when two sides and an angle opposite one of those sides are known. This configuration is called the **SSA case**, meaning Side-Side-Angle.

Unlike the AAS and ASA cases, the SSA information does not always determine a unique triangle. Depending on the given measurements, there may be

1. exactly one triangle,
2. two different triangles, or
3. no triangle.

This uncertainty occurs because the sine function has the same positive value in Quadrants I and II. In particular,

$$\sin\theta = \sin(180^\circ - \theta).$$

Therefore, if the Law of Sines produces an equation of the form

$$\sin\beta = k, \qquad 0 < k < 1,$$

then there are two possible angles between 0° and 180°:

$$\beta_1 = \sin^{-1}(k)$$

and

$$\beta_2 = 180^\circ - \beta_1.$$

However, both values do not necessarily produce valid triangles. Each possible angle must be checked to determine whether the remaining angle is positive.

Suppose α, a, and b are known. The Law of Sines gives

$$\frac{\sin\beta}{b} = \frac{\sin\alpha}{a},$$

so

$$\sin\beta = \frac{b\sin\alpha}{a}.$$

Let

$$k = \frac{b\sin\alpha}{a}.$$

The number of possible triangles depends on the value of k.

- If $k > 1$, then no triangle exists because the sine of an angle cannot exceed 1.
- If $k = 1$, then $\beta = 90^\circ$, so exactly one triangle exists.
- If $0 < k < 1$, then one or two triangles may exist. The principal angle is

  $$\beta_1 = \sin^{-1}(k),$$

  and the second possible angle is

  $$\beta_2 = 180^\circ - \beta_1.$$

  Each angle must be checked together with the known angle α.

For each possible value of β, the remaining angle is

$$\gamma = 180^\circ - \alpha - \beta.$$

A valid triangle requires

$$\gamma > 0^\circ.$$

If $\gamma \le 0^\circ$, that possible value of β must be rejected.

1.7 Example 3: One Triangle

Example 1.4. Solve the triangle if

$$\alpha = 120^\circ, \qquad a = 7, \qquad b = 5.$$

This is an SSA (Side-Side-Angle) problem, so we begin by using the Law of Sines to determine the unknown angle β.

Step 1: Find β.

Using

$$\frac{\sin\beta}{b} = \frac{\sin\alpha}{a},$$

we obtain

$$\sin\beta = \frac{b\sin\alpha}{a}.$$

Substituting the given values,

$$\sin\beta = \frac{5\sin 120^\circ}{7}.$$

Since

$$\sin 120^\circ = \frac{\sqrt{3}}{2},$$

it follows that

$$\sin\beta = \frac{5\sqrt{3}}{14}.$$

Therefore,

$$\boxed{\beta = \sin^{-1}\left(\frac{5\sqrt{3}}{14}\right) \approx 38.2^\circ.}$$

Could there be another solution?

Normally,

$$180^\circ - \beta = 141.8^\circ$$

is another possible angle with the same sine. However,

$$120^\circ + 141.8^\circ > 180^\circ,$$

which is impossible for a triangle.

Hence there is only one possible value of β.

Step 2: Find the third angle.

Using the angle-sum property,

$$\gamma = 180^\circ - 120^\circ - 38.2^\circ \approx 21.8^\circ.$$

Thus,

$$\boxed{\gamma \approx 21.8^\circ.}$$

Step 3: Find side c.
Apply the Law of Sines once more:

$$\frac{c}{\sin\gamma} = \frac{a}{\sin\alpha},$$

so

$$c = \frac{a\sin\gamma}{\sin\alpha}.$$

Substituting the known values,

$$c = \frac{7\sin 21.8^\circ}{\sin 120^\circ} = \frac{14}{\sqrt{3}}\sin 21.8^\circ.$$

Hence,

$$\boxed{c \approx 3.01.}$$

The completed triangle is

$$\boxed{\alpha = 120^\circ, \qquad \beta \approx 38.2^\circ, \qquad \gamma \approx 21.8^\circ,}$$

$$\boxed{a = 7, \qquad b = 5, \qquad c \approx 3.01.}$$

1.8 Example 4: No Triangle Exists

Example 1.5. Determine whether a triangle exists if

$$\gamma = 95^\circ, \qquad c = 3, \qquad a = 4.$$

Again, we begin with the Law of Sines.

$$\frac{\sin\alpha}{a} = \frac{\sin\gamma}{c}.$$

Therefore,

$$\sin\alpha = \frac{a\sin\gamma}{c}.$$

Substituting the given values,

$$\sin\alpha = \frac{4\sin 95^\circ}{3}.$$

Since

$$\sin 95^\circ \approx 0.9962,$$

we obtain

$$\sin\alpha \approx \frac{4(0.9962)}{3} \approx 1.33.$$

However,

$$-1 \leq \sin\theta \leq 1$$

for every angle θ.
Because

$$1.33 > 1,$$

there is no angle whose sine equals 1.33.
Therefore,

$$\boxed{\text{No triangle satisfies the given information.}}$$

This illustrates one of the possible outcomes of the SSA case. Before attempting to compute any angles, it is always important to verify that the value obtained for the sine lies within the interval

$$[-1, 1].$$

1.9 Example 5: Two Triangles

Example 1.6. Find all triangles satisfying

$$\alpha = 30^\circ, \qquad a = 6, \qquad b = 8.$$

This is an SSA problem because two sides and an angle opposite one of those sides are given. Therefore, we must check whether the data produce zero, one, or two triangles.
Step 1: Use the Law of Sines to find β.
From the Law of Sines,

$$\frac{\sin\beta}{b} = \frac{\sin\alpha}{a}.$$

Solving for $\sin\beta$ gives

$$\sin\beta = \frac{b\sin\alpha}{a}.$$

Substituting the given values,

$$\sin\beta = \frac{8\sin 30^\circ}{6}.$$

Since

$$\sin 30^\circ = \frac{1}{2},$$

we obtain

$$\sin\beta = \frac{8(1/2)}{6} = \frac{4}{6} = \frac{2}{3}.$$

Therefore, the principal value is

$$\beta_1 = \sin^{-1}\left(\frac{2}{3}\right) \approx 41.8^\circ.$$

Because sine is positive in both Quadrants I and II, there is a second possible angle:

$$\beta_2 = 180^\circ - \beta_1.$$

Thus,

$$\beta_2 \approx 180^\circ - 41.8^\circ = 138.2^\circ.$$

Both values must now be checked.

1.9.1 First Triangle

For the first triangle,

$$\beta_1 \approx 41.8^\circ.$$

The third angle is

$$\gamma_1 = 180^\circ - \alpha - \beta_1.$$

Therefore,

$$\gamma_1 = 180^\circ - 30^\circ - 41.8^\circ \approx 108.2^\circ.$$

Since $\gamma_1 > 0^\circ$, this produces a valid triangle.
To find c_1, use the Law of Sines:

$$\frac{c_1}{\sin\gamma_1} = \frac{a}{\sin\alpha}.$$

Solving for c_1,

$$c_1 = \frac{a\sin\gamma_1}{\sin\alpha}.$$

Substitute the known values:

$$c_1 = \frac{6\sin 108.2^\circ}{\sin 30^\circ}.$$

Since $\sin 30^\circ = \frac{1}{2}$,

$$c_1 = 12\sin 108.2^\circ.$$

Thus,

$$\boxed{c_1 \approx 11.4.}$$

The first triangle is

$$\boxed{\alpha = 30^\circ, \qquad \beta_1 \approx 41.8^\circ, \qquad \gamma_1 \approx 108.2^\circ,}$$

with side lengths

$$\boxed{a = 6, \qquad b = 8, \qquad c_1 \approx 11.4.}$$

1.9.2 Second Triangle

For the second triangle,

$$\beta_2 \approx 138.2^\circ.$$

The third angle is

$$\gamma_2 = 180^\circ - \alpha - \beta_2.$$

Therefore,

$$\gamma_2 = 180^\circ - 30^\circ - 138.2^\circ \approx 11.8^\circ.$$

Since $\gamma_2 > 0^\circ$, this also produces a valid triangle.
To find c_2, use the Law of Sines:

$$\frac{c_2}{\sin \gamma_2} = \frac{a}{\sin \alpha}.$$

Hence,

$$c_2 = \frac{a \sin \gamma_2}{\sin \alpha}.$$

Substituting the known values,

$$c_2 = \frac{6 \sin 11.8^\circ}{\sin 30^\circ}.$$

Since $\sin 30^\circ = \frac{1}{2}$,

$$c_2 = 12 \sin 11.8^\circ.$$

Thus,

$$\boxed{c_2 \approx 2.5.}$$

The second triangle is

$$\boxed{\alpha = 30^\circ, \qquad \beta_2 \approx 138.2^\circ, \qquad \gamma_2 \approx 11.8^\circ,}$$

with side lengths

$$\boxed{a = 6, \qquad b = 8, \qquad c_2 \approx 2.5.}$$

Therefore, the given measurements determine two distinct triangles.

2. The Law of Cosines

The Law of Sines is most useful when a known side and its opposite angle are available. However, some triangle problems provide three sides or two sides together with the included angle. In such cases, there may be no known opposite side-angle pair, so the Law of Sines cannot be applied immediately.

The **Law of Cosines** relates all three sides of a triangle to one of its angles. It may be viewed as a generalization of the Pythagorean Theorem to triangles that are not necessarily right triangles.

Theorem 2.1 (The Law of Cosines)**.** *Let a triangle have angles α, β, and γ, with opposite sides a, b, and c, respectively. Then*

$$\boxed{a^2 = b^2 + c^2 - 2bc \cos \alpha,}$$

$$\boxed{b^2 = a^2 + c^2 - 2ac\cos\beta,}$$

and

$$\boxed{c^2 = a^2 + b^2 - 2ab\cos\gamma.}$$

Each formula has the same structure. The side being found appears alone on the left, while the angle opposite that side appears in the cosine term.
For example, because side a is opposite angle α, the appropriate formula is

$$a^2 = b^2 + c^2 - 2bc\cos\alpha.$$

Similarly, because side c is opposite angle γ, the corresponding formula is

$$c^2 = a^2 + b^2 - 2ab\cos\gamma.$$

When $\gamma = 90°$, we have

$$\cos 90° = 0.$$

Therefore,

$$c^2 = a^2 + b^2 - 2ab\cos 90° = a^2 + b^2.$$

Thus, for a right triangle, the Law of Cosines reduces to the Pythagorean Theorem.

2.1 The SAS Case

The **SAS case** occurs when two sides of a triangle and the included angle between those sides are known. The included angle is the angle formed by the two known sides.
For example, if sides a and b and the angle γ are known, then γ is the included angle because it lies between sides a and b.
In this situation, the Law of Sines cannot generally be used first because no complete opposite side-angle pair is known. Instead, the Law of Cosines is used to find the third side:

$$c^2 = a^2 + b^2 - 2ab\cos\gamma.$$

Once the third side has been determined, the remaining angles may be found using either the Law of Cosines or the Law of Sines.
The general procedure for solving an SAS triangle is:

1. Identify the two known sides and their included angle.
2. Use the Law of Cosines to find the side opposite the known angle.
3. Use the Law of Cosines or the Law of Sines to determine a second angle.
4. Find the final angle using the angle-sum property.

2.2 Example 6: Solving an SAS Triangle

Example 2.2. Solve the triangle if

$$a = 10, \qquad b = 20, \qquad \gamma = 60°.$$

The known angle γ lies between the known sides a and b, so this is an SAS problem.
Step 1: Find side c.
Because c is opposite γ, the appropriate form of the Law of Cosines is

$$c^2 = a^2 + b^2 - 2ab\cos\gamma.$$

Substitute the given values:

$$c^2 = 10^2 + 20^2 - 2(10)(20)\cos 60^\circ.$$

Since

$$\cos 60^\circ = \frac{1}{2},$$

we obtain

$$\begin{aligned} c^2 &= 100 + 400 - 400\left(\frac{1}{2}\right) \\ &= 500 - 200 \\ &= 300. \end{aligned}$$

Therefore,

$$c = \sqrt{300}.$$

Because a side length must be positive,

$$c = 10\sqrt{3}.$$

Thus,

$$\boxed{c = 10\sqrt{3} \approx 17.32.}$$

Step 2: Find angle α.
We can use the Law of Cosines again. Since side a is opposite α,

$$a^2 = b^2 + c^2 - 2bc\cos\alpha.$$

Solving for $\cos\alpha$ gives

$$2bc\cos\alpha = b^2 + c^2 - a^2,$$

so

$$\cos\alpha = \frac{b^2 + c^2 - a^2}{2bc}.$$

Substitute

$$a = 10, \qquad b = 20, \qquad c = 10\sqrt{3}:$$

$$\cos\alpha = \frac{20^2 + (10\sqrt{3})^2 - 10^2}{2(20)(10\sqrt{3})}.$$

Simplifying the numerator,

$$20^2 + (10\sqrt{3})^2 - 10^2 = 400 + 300 - 100 = 600.$$

Therefore,

$$\cos\alpha = \frac{600}{400\sqrt{3}}.$$

Reducing the fraction,

$$\cos\alpha = \frac{3}{2\sqrt{3}} = \frac{\sqrt{3}}{2}.$$

Hence,

$$\alpha = \cos^{-1}\left(\frac{\sqrt{3}}{2}\right) = 30^\circ.$$

Thus,

$$\boxed{\alpha = 30^\circ.}$$

Step 3: Find angle β.
The interior angles of a triangle sum to 180°, so

$$\beta = 180^\circ - \alpha - \gamma.$$

Substituting the known values,

$$\beta = 180^\circ - 30^\circ - 60^\circ = 90^\circ.$$

Therefore,

$$\boxed{\beta = 90^\circ.}$$

The completed triangle is

$$\boxed{\alpha = 30^\circ, \qquad \beta = 90^\circ, \qquad \gamma = 60^\circ,}$$

with side lengths

$$\boxed{a = 10, \qquad b = 20, \qquad c = 10\sqrt{3}.}$$

This is a 30°–60°–90° triangle. Its side lengths are consistent with the standard ratio

$$1 : \sqrt{3} : 2.$$

Indeed,

$$10 : 10\sqrt{3} : 20 = 1 : \sqrt{3} : 2.$$

2.3 The SSS Case

The **SSS case** occurs when all three side lengths of a triangle are known. Since no angles are initially given, the Law of Sines cannot be used first. The Law of Cosines allows us to determine any one of the angles.
Solving each version of the Law of Cosines for its angle gives

$$\boxed{\alpha = \cos^{-1}\left(\frac{b^2 + c^2 - a^2}{2bc}\right),}$$

$$\boxed{\beta = \cos^{-1}\left(\frac{a^2 + c^2 - b^2}{2ac}\right),}$$

and

$$\boxed{\gamma = \cos^{-1}\left(\frac{a^2 + b^2 - c^2}{2ab}\right).}$$

In practice, it is usually sufficient to use the Law of Cosines to find two angles. The third angle can then be obtained from

$$\alpha + \beta + \gamma = 180^\circ.$$

Before solving an SSS triangle, the side lengths should satisfy the triangle inequalities:

$$a + b > c, \qquad a + c > b, \qquad b + c > a.$$

These inequalities express the fact that the sum of the lengths of any two sides of a triangle must be greater than the length of the remaining side.

2.4 Example 7: Solving an SSS Triangle

Example 2.3. Solve the triangle if

$$a = 5, \qquad b = 7, \qquad c = 8.$$

Since all three side lengths are known, this is an SSS problem. We use the Law of Cosines to determine one angle and then use either the Law of Cosines again or the angle-sum property to find the remaining angles.

Step 1: Find angle γ.

Since side c is opposite angle γ, we use

$$c^2 = a^2 + b^2 - 2ab\cos\gamma.$$

Solving for $\cos\gamma$,

$$\cos\gamma = \frac{a^2 + b^2 - c^2}{2ab}.$$

Substituting the given values,

$$\cos\gamma = \frac{5^2 + 7^2 - 8^2}{2(5)(7)} = \frac{25 + 49 - 64}{70} = \frac{10}{70} = \frac{1}{7}.$$

Therefore,

$$\boxed{\gamma = \cos^{-1}\left(\frac{1}{7}\right) \approx 81.79^\circ.}$$

Step 2: Find angle α.

Since side a is opposite angle α,

$$\cos\alpha = \frac{b^2+c^2-a^2}{2bc}.$$

Substituting the known side lengths,

$$\cos\alpha = \frac{7^2+8^2-5^2}{2(7)(8)} = \frac{49+64-25}{112} = \frac{88}{112} = \frac{11}{14}.$$

Hence,

$$\boxed{\alpha = \cos^{-1}\left(\frac{11}{14}\right) \approx 38.21^\circ.}$$

Step 3: Find angle β.
Using the angle-sum property,

$$\beta = 180^\circ - \alpha - \gamma.$$

Substituting the known values,

$$\beta = 180^\circ - 38.21^\circ - 81.79^\circ = 60.00^\circ.$$

Therefore,

$$\boxed{\beta = 60^\circ.}$$

The completed triangle is

$$\boxed{\alpha \approx 38.21^\circ, \qquad \beta = 60^\circ, \qquad \gamma \approx 81.79^\circ,}$$

with side lengths

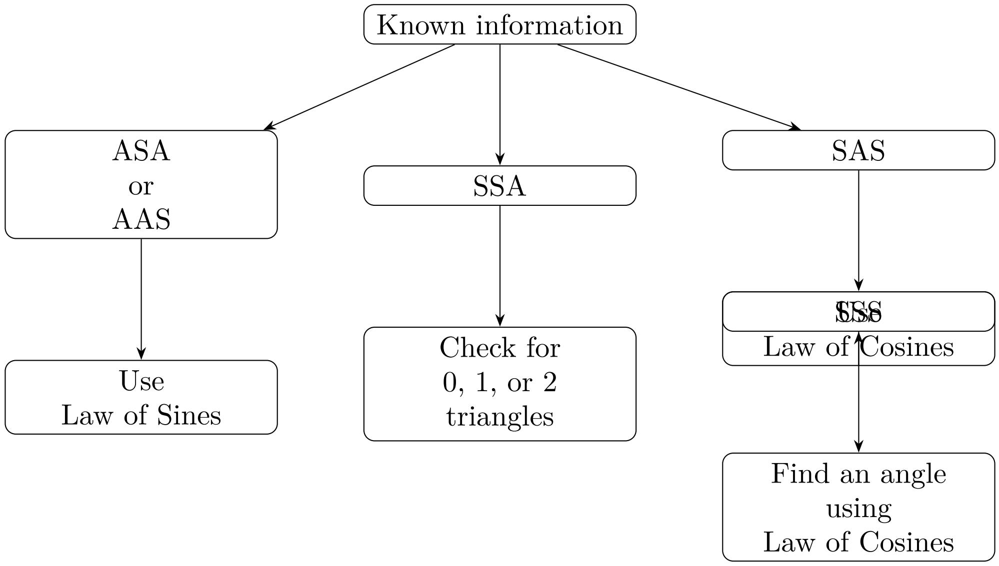
$$\boxed{a = 5, \qquad b = 7, \qquad c = 8.}$$

2.5 Choosing Between the Law of Sines and the Law of Cosines

When solving oblique triangles, the first step is to identify the type of information provided. The choice between the Law of Sines and the Law of Cosines depends entirely on the known sides and angles.

Given Information	**Method**	**Reason**
ASA or AAS	Law of Sines	A known side-angle pair is available.
SSA	Law of Sines	May produce 0, 1, or 2 triangles.
SAS	Law of Cosines	Find the third side first.
SSS	Law of Cosines	Find an angle first.

The following flowchart summarizes this decision process.



The Laws of Sines and Cosines together provide methods for solving any oblique triangle whenever sufficient information is available. In the next section, we shift our attention from solving triangles to developing algebraic relationships among trigonometric functions. These identities will play a central role in simplifying expressions, proving formulas, and solving trigonometric equations later in the chapter.

3. Basic Trigonometric Identities

In previous sections, we used trigonometric functions to solve right triangles and oblique triangles. In many applications, however, it is necessary to simplify complicated trigonometric expressions or verify that two seemingly different expressions are actually equivalent.

A **trigonometric identity** is an equation involving trigonometric functions that is true for every value of the variable for which both sides of the equation are defined.

Unlike a trigonometric equation, which is true only for specific values of the variable, an identity is true for an entire domain.

For example,

$$\sin^2 x + \cos^2 x = 1$$

is a trigonometric identity because it holds for every real number x.

On the other hand,

$$\sin x = \frac{1}{2}$$

is not an identity since it is true only for certain values of x.

Throughout this section, we develop several important identities that will be used repeatedly in algebraic simplification, calculus, physics, engineering, and many other areas of mathematics.

3.1 Reciprocal Identities

The six trigonometric functions occur naturally in reciprocal pairs. These relationships follow directly from the definitions of the trigonometric functions.

Definition 3.1 (Reciprocal Identities)**.** The reciprocal identities are

$$\boxed{\csc x = \frac{1}{\sin x},}$$

$$\boxed{\sec x = \frac{1}{\cos x},}$$

and

$$\boxed{\cot x = \frac{1}{\tan x}.}$$

Equivalently,

$$\boxed{\sin x = \frac{1}{\csc x}, \qquad \cos x = \frac{1}{\sec x}, \qquad \tan x = \frac{1}{\cot x}.}$$

These identities are valid whenever the denominators are nonzero.
For example,

$$\sec x = \frac{1}{\cos x}$$

is undefined whenever

$$\cos x = 0.$$

Similarly,

$$\csc x = \frac{1}{\sin x}$$

is undefined whenever

$$\sin x = 0.$$

3.2 Quotient Identities

The tangent and cotangent functions may be written as quotients of the sine and cosine functions.

Definition 3.2 (Quotient Identities)**.**

$$\boxed{\tan x = \frac{\sin x}{\cos x},}$$

and

$$\boxed{\cot x = \frac{\cos x}{\sin x}.}$$

These identities are extremely useful because they allow every occurrence of tangent or cotangent to be rewritten entirely in terms of sine and cosine.
For example,

$$\tan^2 x = \left(\frac{\sin x}{\cos x}\right)^2 = \frac{\sin^2 x}{\cos^2 x}.$$

Likewise,

$$\cot^2 x = \frac{\cos^2 x}{\sin^2 x}.$$

3.3 Example 1

Example 3.3. Rewrite

$$\sec x \tan x$$

using only sine and cosine.

Using the reciprocal identity,

$$\sec x = \frac{1}{\cos x},$$

and the quotient identity,

$$\tan x = \frac{\sin x}{\cos x},$$

we obtain

$$\sec x \tan x = \left(\frac{1}{\cos x}\right)\left(\frac{\sin x}{\cos x}\right).$$

Multiplying the fractions gives

$$\boxed{\sec x \tan x = \frac{\sin x}{\cos^2 x}.}$$

3.4 Pythagorean Identities

The most important trigonometric identities arise from the Pythagorean Theorem.
Consider a point

$$(x, y)$$

lying on the unit circle.
Since every point on the unit circle satisfies

$$x^2 + y^2 = 1,$$

and

$$x = \cos\theta, \qquad y = \sin\theta,$$

substituting these expressions gives

$$\boxed{\sin^2\theta + \cos^2\theta = 1.}$$

This identity is called the **fundamental Pythagorean identity** because all other Pythagorean identities can be derived from it.
Dividing both sides by $\cos^2\theta$ gives

$$\frac{\sin^2\theta}{\cos^2\theta} + 1 = \frac{1}{\cos^2\theta},$$

or

$$\boxed{1 + \tan^2\theta = \sec^2\theta.}$$

Similarly, dividing by $\sin^2\theta$ gives

$$1 + \frac{\cos^2\theta}{\sin^2\theta} = \frac{1}{\sin^2\theta},$$

which becomes

$$\boxed{1 + \cot^2\theta = \csc^2\theta.}$$

Therefore, the three Pythagorean identities are

$$\boxed{\sin^2 x + \cos^2 x = 1,}$$

$$\boxed{1 + \tan^2 x = \sec^2 x,}$$

and

$$\boxed{1 + \cot^2 x = \csc^2 x.}$$

These identities are among the most frequently used formulas in all of trigonometry.

3.5 Example 2

Example 3.4. Simplify

$$1 + \tan^2 x.$$

Using the second Pythagorean identity,

$$1 + \tan^2 x = \sec^2 x.$$

Therefore,

$$\boxed{1 + \tan^2 x = \sec^2 x.}$$

3.6 Example 3

Example 3.5. Simplify

$$\csc^2 x - 1.$$

From the identity

$$1 + \cot^2 x = \csc^2 x,$$

subtracting 1 from both sides gives

$$\boxed{\csc^2 x - 1 = \cot^2 x.}$$

3.7 Even and Odd Identities

Many trigonometric functions possess important symmetry properties. These properties allow us to evaluate trigonometric functions of negative angles without referring to a graph or the unit circle.

Recall that replacing an angle θ by $-\theta$ corresponds to reflecting the terminal side of the angle across the x-axis. Some functions remain unchanged under this reflection, while others change their sign.

3.7.1 Even Functions

A function is called **even** if

$$f(-x) = f(x)$$

for every value of x in its domain.
Geometrically, the graph of an even function is symmetric about the y-axis.
Among the trigonometric functions, cosine and secant are even functions.

Theorem 3.6 (Even Identities)**.**

$$\boxed{\cos(-x) = \cos x,}$$

$$\boxed{\sec(-x) = \sec x.}$$

These identities indicate that changing the sign of the angle does not change the value of either cosine or secant.

3.7.2 Odd Functions

A function is called **odd** if

$$f(-x) = -f(x)$$

for every value of x in its domain.
The graph of an odd function is symmetric with respect to the origin.
The remaining four trigonometric functions are odd.

Theorem 3.7 (Odd Identities)**.**

$$\boxed{\sin(-x) = -\sin x,}$$

$$\boxed{\tan(-x) = -\tan x,}$$

$$\boxed{\csc(-x) = -\csc x,}$$

and

$$\boxed{\cot(-x) = -\cot x.}$$

Whenever the angle is replaced by its negative, the value of each odd function changes sign.

3.8 Example 4

Example 3.8. Evaluate

$$\sin(-45^\circ).$$

Since sine is an odd function,

$$\sin(-\theta) = -\sin\theta.$$

Therefore,

$$\sin(-45^\circ) = -\sin 45^\circ.$$

Because

$$\sin 45^\circ = \frac{\sqrt{2}}{2},$$

we conclude that

$$\boxed{\sin(-45^\circ) = -\frac{\sqrt{2}}{2}.}$$

3.9 Example 5

Example 3.9. Evaluate

$$\cos(-120^\circ).$$

Cosine is an even function, so

$$\cos(-\theta) = \cos\theta.$$

Hence,

$$\cos(-120^\circ) = \cos 120^\circ.$$

Since

$$\cos 120^\circ = -\frac{1}{2},$$

we obtain

$$\boxed{\cos(-120^\circ) = -\frac{1}{2}.}$$

3.10 Summary of the Fundamental Identities

The reciprocal, quotient, Pythagorean, and even–odd identities form the foundation of trigonometric algebra. These identities should be memorized because they are used repeatedly when simplifying expressions, proving identities, and solving trigonometric equations.

Identity Type	**Formula**
Reciprocal	$\sin x = \frac{1}{\csc x},\ \cos x = \frac{1}{\sec x},\ \tan x = \frac{1}{\cot x}$
	$\csc x = \frac{1}{\sin x},\ \sec x = \frac{1}{\cos x},\ \cot x = \frac{1}{\tan x}$
Quotient	$\tan x = \frac{\sin x}{\cos x}, \qquad \cot x = \frac{\cos x}{\sin x}$
Pythagorean	$\sin^2 x + \cos^2 x = 1$
	$1 + \tan^2 x = \sec^2 x$
	$1 + \cot^2 x = \csc^2 x$
Even	$\cos(-x) = \cos x, \qquad \sec(-x) = \sec x$
Odd	$\sin(-x) = -\sin x,\ \tan(-x) = -\tan x,\ \csc(-x) = -\csc x,\ \cot(-x) = -\cot x$

These identities will be used extensively in the next section, where we derive formulas for the sine, cosine, and tangent of the sum or difference of two angles.

4. Addition and Subtraction Formulas

Many trigonometric problems involve angles that are written as the sum or difference of two simpler angles. For example,

$$75^\circ = 45^\circ + 30^\circ,$$

and

$$15^\circ = 45^\circ - 30^\circ.$$

Neither 75° nor 15° is a special angle whose exact trigonometric values are usually memorized. However, by expressing these angles as sums or differences of familiar angles, we can compute their exact trigonometric values.

The addition and subtraction formulas provide relationships between the trigonometric functions of a combined angle and the corresponding functions of its individual parts. These formulas are among the most important identities in trigonometry and form the basis for many results in calculus, differential equations, and Fourier analysis.

4.1 The Addition and Subtraction Formulas

Theorem 4.1 (Addition and Subtraction Formulas)**.** *For any real numbers α and β,*

$$\boxed{\sin(\alpha + \beta) = \sin\alpha\cos\beta + \cos\alpha\sin\beta,}$$

$$\boxed{\sin(\alpha - \beta) = \sin\alpha\cos\beta - \cos\alpha\sin\beta,}$$

$$\boxed{\cos(\alpha + \beta) = \cos\alpha\cos\beta - \sin\alpha\sin\beta,}$$

$$\boxed{\cos(\alpha - \beta) = \cos\alpha\cos\beta + \sin\alpha\sin\beta,}$$

and

$$\boxed{\tan(\alpha \pm \beta) = \frac{\tan\alpha \pm \tan\beta}{1 \mp \tan\alpha\tan\beta},}$$

provided the denominator is nonzero.

Notice that the sine formulas contain opposite signs, whereas the cosine formulas contain the opposite sign pattern. A convenient way to remember these identities is

$$\sin(\alpha \pm \beta) = \sin\alpha\cos\beta \pm \cos\alpha\sin\beta,$$

$$\cos(\alpha \pm \beta) = \cos\alpha\cos\beta \mp \sin\alpha\sin\beta.$$

4.2 Example 1

Example 4.2. Find the exact value of

$$\sin 75^\circ.$$

Since

$$75^\circ = 45^\circ + 30^\circ,$$

we use the sine addition formula.

$$\sin 75^\circ = \sin(45^\circ + 30^\circ).$$

Applying the identity,

$$\sin 75^\circ = \sin 45^\circ \cos 30^\circ + \cos 45^\circ \sin 30^\circ.$$

Using the exact values

$$\sin 45^\circ = \cos 45^\circ = \frac{\sqrt{2}}{2},$$

$$\cos 30^\circ = \frac{\sqrt{3}}{2}, \qquad \sin 30^\circ = \frac{1}{2},$$

gives

$$\begin{aligned} \sin 75^\circ &= \frac{\sqrt{2}}{2} \cdot \frac{\sqrt{3}}{2} + \frac{\sqrt{2}}{2} \cdot \frac{1}{2} \\ &= \frac{\sqrt{6}+\sqrt{2}}{4}. \end{aligned}$$

Therefore,

$$\boxed{\sin 75^\circ = \frac{\sqrt{6}+\sqrt{2}}{4}.}$$

4.3 Example 2

Example 4.3. Find the exact value of

$$\cos 15^\circ.$$

Observe that

$$15^\circ = 45^\circ - 30^\circ.$$

Using the cosine subtraction formula,

$$\cos 15^\circ = \cos 45^\circ \cos 30^\circ + \sin 45^\circ \sin 30^\circ.$$

Substituting the exact values,

$$\begin{aligned} \cos 15^\circ &= \frac{\sqrt{2}}{2} \cdot \frac{\sqrt{3}}{2} + \frac{\sqrt{2}}{2} \cdot \frac{1}{2} \\ &= \frac{\sqrt{6}+\sqrt{2}}{4}. \end{aligned}$$

Hence,

$$\boxed{\cos 15^\circ = \frac{\sqrt{6}+\sqrt{2}}{4}.}$$

4.4 Example 3

Example 4.4. Find the exact value of

$$\cos 75^\circ.$$

Since

$$75^\circ = 45^\circ + 30^\circ,$$

we apply the cosine addition formula.

$$\cos 75^\circ = \cos 45^\circ \cos 30^\circ - \sin 45^\circ \sin 30^\circ.$$

Substituting the known values,

$$\begin{aligned}\cos 75^\circ &= \frac{\sqrt{2}}{2} \cdot \frac{\sqrt{3}}{2} - \frac{\sqrt{2}}{2} \cdot \frac{1}{2} \\ &= \frac{\sqrt{6} - \sqrt{2}}{4}.\end{aligned}$$

Therefore,

$$\boxed{\cos 75^\circ = \frac{\sqrt{6} - \sqrt{2}}{4}.}$$

4.5 Example 4

Example 4.5. Find the exact value of

$$\tan 75^\circ.$$

Notice that

$$75^\circ = 45^\circ + 30^\circ.$$

Therefore, we apply the tangent addition formula,

$$\tan(\alpha + \beta) = \frac{\tan \alpha + \tan \beta}{1 - \tan \alpha \tan \beta}.$$

Substituting $\alpha = 45^\circ$ and $\beta = 30^\circ$,

$$\tan 75^\circ = \frac{\tan 45^\circ + \tan 30^\circ}{1 - \tan 45^\circ \tan 30^\circ}.$$

Using the exact values

$$\tan 45^\circ = 1, \qquad \tan 30^\circ = \frac{1}{\sqrt{3}},$$

gives

$$\tan 75^\circ = \frac{1 + \frac{1}{\sqrt{3}}}{1 - \frac{1}{\sqrt{3}}}.$$

Multiply both the numerator and denominator by $\sqrt{3}$:

$$\tan 75^\circ = \frac{\sqrt{3} + 1}{\sqrt{3} - 1}.$$

To simplify further, multiply by the conjugate:

$$\begin{aligned}\tan 75^\circ &= \frac{\sqrt{3}+1}{\sqrt{3}-1}\cdot\frac{\sqrt{3}+1}{\sqrt{3}+1}\\ &= \frac{(\sqrt{3}+1)^2}{3-1}\\ &= \frac{3+2\sqrt{3}+1}{2}\\ &= 2+\sqrt{3}.\end{aligned}$$

Hence,

$$\boxed{\tan 75^\circ = 2+\sqrt{3}.}$$

4.6 Example 5

Example 4.6. Find the exact value of

$$\tan 15^\circ.$$

Since

$$15^\circ = 45^\circ - 30^\circ,$$

we use the tangent subtraction formula,

$$\tan(\alpha-\beta) = \frac{\tan\alpha - \tan\beta}{1+\tan\alpha\tan\beta}.$$

Substituting the known values,

$$\tan 15^\circ = \frac{1-\frac{1}{\sqrt{3}}}{1+\frac{1}{\sqrt{3}}}.$$

Multiply the numerator and denominator by $\sqrt{3}$:

$$\tan 15^\circ = \frac{\sqrt{3}-1}{\sqrt{3}+1}.$$

Now multiply by the conjugate of the denominator:

$$\begin{aligned}\tan 15^\circ &= \frac{\sqrt{3}-1}{\sqrt{3}+1}\cdot\frac{\sqrt{3}-1}{\sqrt{3}-1}\\ &= \frac{(\sqrt{3}-1)^2}{3-1}\\ &= \frac{3-2\sqrt{3}+1}{2}\\ &= 2-\sqrt{3}.\end{aligned}$$

Therefore,

$$\boxed{\tan 15^\circ = 2 - \sqrt{3}.}$$

4.7 Using the Addition and Subtraction Formulas

The addition and subtraction formulas are especially useful for finding exact values of trigonometric functions at angles that are not among the standard special angles. By expressing an angle as the sum or difference of two familiar angles, we can evaluate the desired trigonometric function using only known exact values.

The general strategy is as follows.

1. Rewrite the given angle as the sum or difference of two special angles such as

$$30^\circ,\ 45^\circ,\ 60^\circ,\ 90^\circ.$$

2. Select the appropriate addition or subtraction formula.
3. Substitute the exact trigonometric values of the special angles.
4. Simplify the resulting expression carefully, rationalizing the denominator whenever necessary.

Some common angle decompositions include

$$75^\circ = 45^\circ + 30^\circ,$$

$$15^\circ = 45^\circ - 30^\circ,$$

$$105^\circ = 60^\circ + 45^\circ,$$

and

$$135^\circ = 90^\circ + 45^\circ.$$

Recognizing these relationships makes it possible to compute exact values without using a calculator.

5. Double-Angle Formulas

The addition formulas become particularly useful when the two angles are equal. By setting

$$\alpha = \beta = \theta,$$

the addition formulas simplify to identities involving the angle 2θ. These identities are called the **double-angle formulas**.

They provide convenient ways to rewrite trigonometric expressions and play an important role in integration, differential equations, Fourier analysis, and many other applications.

In the following section, we derive formulas for

$$\sin(2\theta), \qquad \cos(2\theta), \qquad \tan(2\theta),$$

and learn how to use them to evaluate trigonometric expressions and solve equations.

5.1 Double-Angle Formulas

In the previous section, we introduced formulas for the sine, cosine, and tangent of the sum or difference of two angles. A particularly important special case occurs when the two angles are equal.

If we let

$$\alpha = \beta = \theta,$$

then

$$\alpha + \beta = 2\theta.$$

Substituting this into the addition formulas produces identities for the trigonometric functions of twice an angle. These identities are called the **double-angle formulas**.
Double-angle formulas are useful for simplifying trigonometric expressions, evaluating exact values, proving identities, solving equations, and studying periodic phenomena in mathematics and physics.

5.2 The Double-Angle Formulas

Starting with the sine addition formula,

$$\sin(\alpha + \beta) = \sin\alpha\cos\beta + \cos\alpha\sin\beta,$$

and setting

$$\alpha = \beta = \theta,$$

gives

$$\boxed{\sin(2\theta) = 2\sin\theta\cos\theta.}$$

Similarly, applying the cosine addition formula,

$$\cos(\alpha + \beta) = \cos\alpha\cos\beta - \sin\alpha\sin\beta,$$

yields

$$\boxed{\cos(2\theta) = \cos^2\theta - \sin^2\theta.}$$

Using the tangent addition formula,

$$\tan(\alpha + \beta) = \frac{\tan\alpha + \tan\beta}{1 - \tan\alpha\tan\beta},$$

gives

$$\boxed{\tan(2\theta) = \frac{2\tan\theta}{1 - \tan^2\theta},}$$

provided

$$1 - \tan^2\theta \neq 0.$$

5.3 Equivalent Forms of the Cosine Double-Angle Formula

The identity

$$\cos(2\theta) = \cos^2\theta - \sin^2\theta$$

can be rewritten in two additional forms by using the Pythagorean identity

$$\sin^2\theta + \cos^2\theta = 1.$$

First Form. Replace

$$\sin^2\theta = 1 - \cos^2\theta.$$

Then

$$\begin{aligned}\cos(2\theta) &= \cos^2\theta - (1 - \cos^2\theta)\\ &= 2\cos^2\theta - 1.\end{aligned}$$

Thus,

$$\boxed{\cos(2\theta) = 2\cos^2\theta - 1.}$$

Second Form. Replace

$$\cos^2\theta = 1 - \sin^2\theta.$$

Then

$$\begin{aligned}\cos(2\theta) &= (1 - \sin^2\theta) - \sin^2\theta\\ &= 1 - 2\sin^2\theta.\end{aligned}$$

Hence,

$$\boxed{\cos(2\theta) = 1 - 2\sin^2\theta.}$$

Therefore, the cosine double-angle identity has three equivalent forms:

$$\boxed{\cos(2\theta) = \cos^2\theta - \sin^2\theta,}$$

$$\boxed{\cos(2\theta) = 2\cos^2\theta - 1,}$$

and

$$\boxed{\cos(2\theta) = 1 - 2\sin^2\theta.}$$

The choice of which form to use depends on the information given in the problem.

5.4 Example 1

Example 5.1. Find the exact value of

$$\sin 60^\circ.$$

Notice that

$$60^\circ = 2(30^\circ).$$

Using the sine double-angle formula,

$$\sin(2\theta) = 2\sin\theta\cos\theta,$$

with

$$\theta = 30^\circ,$$

we obtain

$$\begin{aligned}\sin 60^\circ &= 2\sin 30^\circ \cos 30^\circ \\ &= 2\left(\frac{1}{2}\right)\left(\frac{\sqrt{3}}{2}\right) \\ &= \frac{\sqrt{3}}{2}.\end{aligned}$$

Therefore,

$$\boxed{\sin 60^\circ = \frac{\sqrt{3}}{2}.}$$

5.5 Example 2

Example 5.2. Find the exact value of

$$\cos 60^\circ.$$

Since

$$60^\circ = 2(30^\circ),$$

we use

$$\cos(2\theta) = 2\cos^2\theta - 1.$$

Substituting

$$\theta = 30^\circ,$$

gives

$$\begin{aligned}\cos 60^\circ &= 2\left(\frac{\sqrt{3}}{2}\right)^2 - 1 \\ &= 2\left(\frac{3}{4}\right) - 1 \\ &= \frac{3}{2} - 1 \\ &= \frac{1}{2}.\end{aligned}$$

Hence,

$$\boxed{\cos 60^\circ = \frac{1}{2}.}$$

5.6 Example 3

Example 5.3. Find the exact value of

$$\tan 60^\circ.$$

Since

$$60^\circ = 2(30^\circ),$$

the tangent double-angle formula gives

$$\tan 60^\circ = \frac{2\tan 30^\circ}{1-\tan^2 30^\circ}.$$

Using

$$\tan 30^\circ = \frac{1}{\sqrt{3}},$$

we have

$$\begin{aligned}\tan 60^\circ &= \frac{\frac{2}{\sqrt{3}}}{1-\frac{1}{3}} \\ &= \frac{\frac{2}{\sqrt{3}}}{\frac{2}{3}} \\ &= \frac{3}{\sqrt{3}} \\ &= \sqrt{3}.\end{aligned}$$

Therefore,

$$\boxed{\tan 60^\circ = \sqrt{3}.}$$

5.7 Choosing the Appropriate Double-Angle Formula

The double-angle identities are not all used in the same way. The most appropriate formula depends on the information provided in the problem.

- If both $\sin\theta$ and $\cos\theta$ are known, use

$$\sin(2\theta) = 2\sin\theta\cos\theta.$$

- If $\cos\theta$ is known, it is often convenient to use

$$\cos(2\theta) = 2\cos^2\theta - 1.$$

- If $\sin\theta$ is known, use

$$\cos(2\theta) = 1 - 2\sin^2\theta.$$

- If both $\sin\theta$ and $\cos\theta$ are known, either form

$$\cos(2\theta) = \cos^2\theta - \sin^2\theta$$

  or one of its equivalent identities may be used.
- If $\tan\theta$ is given, use

$$\tan(2\theta) = \frac{2\tan\theta}{1-\tan^2\theta}.$$

Choosing the correct identity often makes the computation much shorter.

5.8 Example 4

Example 5.4. Suppose

$$\sin\theta = \frac{3}{5},$$

where

$$0^\circ < \theta < 90^\circ.$$

Find the exact value of

$$\cos(2\theta).$$

Since $\sin\theta$ is given, the most convenient identity is

$$\cos(2\theta) = 1 - 2\sin^2\theta.$$

Substituting

$$\sin\theta = \frac{3}{5},$$

gives

$$\begin{aligned}\cos(2\theta) &= 1 - 2\left(\frac{3}{5}\right)^2 \\ &= 1 - \frac{18}{25} \\ &= \frac{7}{25}.\end{aligned}$$

Therefore,

$$\boxed{\cos(2\theta) = \frac{7}{25}.}$$

5.9 Example 5

Example 5.5. Suppose

$$\cos\theta = \frac{4}{5},$$

where

$$0^\circ < \theta < 90^\circ.$$

Find the exact value of

$$\cos(2\theta).$$

Since $\cos\theta$ is known, we use

$$\cos(2\theta) = 2\cos^2\theta - 1.$$

Substituting,

$$\begin{aligned}\cos(2\theta) &= 2\left(\frac{4}{5}\right)^2 - 1\\ &= 2\left(\frac{16}{25}\right) - 1\\ &= \frac{32}{25} - 1\\ &= \frac{7}{25}.\end{aligned}$$

Hence,

$$\boxed{\cos(2\theta) = \frac{7}{25}.}$$

Notice that this agrees with the previous example since

$$\sin^2\theta + \cos^2\theta = 1.$$

5.10 Example 6

Example 5.6. Suppose

$$\tan\theta = \frac{1}{2}.$$

Find the exact value of

$$\tan(2\theta).$$

Apply the tangent double-angle identity,

$$\tan(2\theta) = \frac{2\tan\theta}{1-\tan^2\theta}.$$

Substituting

$$\tan\theta = \frac{1}{2},$$

gives

$$\begin{aligned}\tan(2\theta) &= \frac{2\left(\frac{1}{2}\right)}{1-\left(\frac{1}{2}\right)^2}\\ &= \frac{1}{1-\frac{1}{4}}\\ &= \frac{1}{\frac{3}{4}}\\ &= \frac{4}{3}.\end{aligned}$$

Therefore,

$$\boxed{\tan(2\theta) = \frac{4}{3}.}$$

5.11 Example 7

Example 5.7. Verify the identity

$$2\sin x\cos x = \sin(2x).$$

Beginning with the left-hand side,

$$2\sin x\cos x,$$

we recognize this as the sine double-angle formula,

$$\sin(2x) = 2\sin x\cos x.$$

Therefore,

$$2\sin x\cos x = \sin(2x),$$

which agrees with the right-hand side.
Hence,

$$\boxed{2\sin x\cos x = \sin(2x).}$$

5.12 Applications of the Double-Angle Formulas

The double-angle identities are frequently used to

- evaluate exact trigonometric values,
- simplify algebraic expressions,
- verify trigonometric identities,
- rewrite expressions in equivalent forms,
- solve trigonometric equations, and
- prepare for the half-angle identities.

The half-angle identities are obtained by solving the cosine double-angle formulas for either

$$\sin^2\theta$$

or

$$\cos^2\theta,$$

and then replacing θ by $\dfrac{\theta}{2}$.

6. Half-Angle Formulas

The double-angle identities express the trigonometric functions of 2θ in terms of the functions of θ. By reversing this process, we obtain formulas for the trigonometric functions of one-half of an angle. These identities are known as the **half-angle formulas**.

Half-angle formulas are useful when evaluating exact trigonometric values, simplifying expressions, and solving trigonometric equations. They are also widely used in calculus, physics, and engineering.

6.1 Derivation of the Half-Angle Formulas

We begin with the double-angle identity

$$\cos(2\theta) = 1 - 2\sin^2\theta.$$

Solving for $\sin^2\theta$,

$$\begin{aligned} 2\sin^2\theta &= 1-\cos(2\theta), \\ \sin^2\theta &= \frac{1-\cos(2\theta)}{2}. \end{aligned}$$

Now replace θ by $\dfrac{\theta}{2}$:

$$\boxed{\sin^2\left(\frac{\theta}{2}\right) = \frac{1-\cos\theta}{2}.}$$

Taking the square root gives

$$\boxed{\sin\left(\frac{\theta}{2}\right) = \pm\sqrt{\frac{1-\cos\theta}{2}}.}$$

The sign depends on the quadrant in which

$$\frac{\theta}{2}$$

lies.

Similarly, begin with

$$\cos(2\theta) = 2\cos^2\theta - 1.$$

Solving for $\cos^2\theta$,

$$\begin{aligned} 2\cos^2\theta &= 1+\cos(2\theta), \\ \cos^2\theta &= \frac{1+\cos(2\theta)}{2}. \end{aligned}$$

Replacing θ with

$$\frac{\theta}{2},$$

gives

$$\boxed{\cos^2\left(\frac{\theta}{2}\right) = \frac{1+\cos\theta}{2}.}$$

Hence,

$$\boxed{\cos\left(\frac{\theta}{2}\right) = \pm\sqrt{\frac{1+\cos\theta}{2}}.}$$

Again, the correct sign depends on the quadrant containing

$$\frac{\theta}{2}.$$

6.2 The Tangent Half-Angle Formula

Using

$$\tan\theta = \frac{\sin\theta}{\cos\theta},$$

together with the half-angle identities for sine and cosine, we obtain

$$\boxed{\tan\left(\frac{\theta}{2}\right) = \pm\sqrt{\frac{1-\cos\theta}{1+\cos\theta}},}$$

provided the denominator is nonzero.
Another useful equivalent form is

$$\boxed{\tan\left(\frac{\theta}{2}\right) = \frac{\sin\theta}{1+\cos\theta},}$$

and, whenever

$$1-\cos\theta \neq 0,$$

$$\boxed{\tan\left(\frac{\theta}{2}\right) = \frac{1-\cos\theta}{\sin\theta}.}$$

All three formulas are algebraically equivalent, and the choice of which one to use depends on the information provided in the problem.

6.3 Example 1

Example 6.1. Find the exact value of

$$\sin 75^\circ.$$

Observe that

$$75^\circ = \frac{150^\circ}{2}.$$

Using the half-angle formula,

$$\sin\left(\frac{\theta}{2}\right) = \pm\sqrt{\frac{1-\cos\theta}{2}},$$

with

$$\theta = 150^\circ,$$

gives

$$\sin 75^\circ = \sqrt{\frac{1-\cos 150^\circ}{2}},$$

since 75° lies in Quadrant I, where sine is positive.
Because

$$\cos 150^\circ = -\frac{\sqrt{3}}{2},$$

we obtain

$$\begin{aligned}\sin 75^\circ &= \sqrt{\frac{1+\frac{\sqrt{3}}{2}}{2}}\\ &= \sqrt{\frac{2+\sqrt{3}}{4}}\\ &= \frac{\sqrt{2+\sqrt{3}}}{2}.\end{aligned}$$

Therefore,

$$\boxed{\sin 75^\circ = \frac{\sqrt{2+\sqrt{3}}}{2}.}$$

6.4 Example 2

Example 6.2. Find the exact value of

$$\cos 75^\circ.$$

Since

$$75^\circ = \frac{150^\circ}{2},$$

the cosine half-angle identity gives

$$\cos 75^\circ = \sqrt{\frac{1+\cos 150^\circ}{2}},$$

because cosine is positive in Quadrant I.
Substituting

$$\cos 150^\circ = -\frac{\sqrt{3}}{2},$$

yields

$$\begin{aligned}\cos 75^\circ &= \sqrt{\frac{1-\frac{\sqrt{3}}{2}}{2}}\\ &= \sqrt{\frac{2-\sqrt{3}}{4}}\\ &= \frac{\sqrt{2-\sqrt{3}}}{2}.\end{aligned}$$

Hence,

$$\boxed{\cos 75^\circ = \frac{\sqrt{2-\sqrt{3}}}{2}.}$$

6.5 Choosing the Correct Sign

Unlike the double-angle identities, the half-angle formulas involve a square root. Every square root has both a positive and a negative value, so it is necessary to determine the correct sign by locating the angle

$$\frac{\theta}{2}$$

on the coordinate plane.
For example,

- if $\frac{\theta}{2}$ lies in Quadrant I, both sine and cosine are positive;
- if $\frac{\theta}{2}$ lies in Quadrant II, sine is positive and cosine is negative;
- if $\frac{\theta}{2}$ lies in Quadrant III, both sine and cosine are negative;
- if $\frac{\theta}{2}$ lies in Quadrant IV, sine is negative and cosine is positive.

Therefore, before applying a half-angle formula, always determine the quadrant of the half-angle to choose the appropriate sign.

7. Trigonometric Equations

A **trigonometric equation** is an equation that contains one or more trigonometric functions of an unknown angle or variable. Unlike trigonometric identities, which are true for every value in their domain, trigonometric equations are satisfied only for specific values of the variable.
For example,

$$\sin^2 x + \cos^2 x = 1$$

is an identity because it is true for every real number x, whereas

$$\sin x = \frac{1}{2}$$

is a trigonometric equation because it is true only for certain values of x.
The objective when solving a trigonometric equation is to determine all values of the variable that satisfy the equation over a specified interval or over all real numbers.

7.1 General Strategy

Many trigonometric equations can be solved using the following procedure.

1. Simplify both sides of the equation whenever possible.
2. Rewrite the equation so that only one trigonometric function appears.
3. Solve the resulting basic trigonometric equation.
4. Determine every solution in the required interval.
5. Verify each solution in the original equation whenever necessary.

When solving on an interval such as

$$0^\circ \leq x < 360^\circ,$$

all solutions within that interval must be included.

7.2 Equations Involving Sine

Suppose we wish to solve

$$\sin x = k,$$

where

$$-1 \leq k \leq 1.$$

First determine the principal angle

$$\alpha = \sin^{-1}(k).$$

Since sine is positive in Quadrants I and II and negative in Quadrants III and IV, the remaining solution is determined by the sign of k.

7.3 Example 1

Example 7.1. Solve

$$\sin x = \frac{1}{2}, \qquad 0^\circ \leq x < 360^\circ.$$

The reference angle is

$$30^\circ,$$

because

$$\sin 30^\circ = \frac{1}{2}.$$

Since sine is positive in Quadrants I and II,

$$x = 30^\circ$$

or

$$x = 180^\circ - 30^\circ = 150^\circ.$$

Therefore,

$$\boxed{x = 30^\circ,\ 150^\circ.}$$

7.4 Equations Involving Cosine

For equations of the form

$$\cos x = k,$$

the reference angle is

$$\alpha = \cos^{-1}(k).$$

Cosine is positive in Quadrants I and IV and negative in Quadrants II and III.

7.5 Example 2

Example 7.2. Solve

$$\cos x = -\frac{1}{2}, \qquad 0^\circ \le x < 360^\circ.$$

The reference angle is

$$60^\circ.$$

Since cosine is negative in Quadrants II and III,

$$x = 180^\circ - 60^\circ = 120^\circ,$$

or

$$x = 180^\circ + 60^\circ = 240^\circ.$$

Hence,

$$\boxed{x = 120^\circ,\ 240^\circ.}$$

7.6 Equations Involving Tangent

For equations of the form

$$\tan x = k,$$

the principal solution is

$$\alpha = \tan^{-1}(k).$$

Since tangent has period

$$180^\circ,$$

the solutions differ by integer multiples of

$$180^\circ.$$

7.7 Example 3

Example 7.3. Solve

$$\tan x = 1, \qquad 0^\circ \le x < 360^\circ.$$

The reference angle is

$$45^\circ.$$

Since tangent is positive in Quadrants I and III,

$$x = 45^\circ$$

or

$$x = 225^\circ.$$

Therefore,

$$\boxed{x = 45^\circ,\ 225^\circ.}$$

7.8 Equations Requiring Algebraic Manipulation

Many trigonometric equations must first be rewritten before the basic trigonometric solutions can be applied.

Typical algebraic techniques include

- factoring,
- using trigonometric identities,
- completing the square,
- applying double-angle or half-angle formulas,
- isolating a single trigonometric function.

7.9 Example 4

Example 7.4. Solve

$$2\sin x - 1 = 0, \qquad 0^\circ \le x < 360^\circ.$$

First isolate the sine function.

$$2\sin x = 1,$$

so

$$\sin x = \frac{1}{2}.$$

From Example 1,

$$\boxed{x = 30^\circ,\ 150^\circ.}$$

7.10 Example 5

Example 7.5. Solve

$$2\cos^2 x - 1 = 0, \qquad 0^\circ \le x < 360^\circ.$$

First isolate the cosine term.

$$2\cos^2 x = 1,$$

so

$$\cos^2 x = \frac{1}{2}.$$

Taking square roots,

$$\cos x = \pm\frac{\sqrt{2}}{2}.$$

When

$$\cos x = \frac{\sqrt{2}}{2},$$

the solutions are

$$45^\circ,\ 315^\circ.$$

When

$$\cos x = -\frac{\sqrt{2}}{2},$$

the solutions are

$$135^\circ,\ 225^\circ.$$

Therefore,

$$\boxed{x = 45^\circ,\ 135^\circ,\ 225^\circ,\ 315^\circ.}$$

7.11 Example 6

Example 7.6. Solve

$$\sin(2x) = \frac{\sqrt{3}}{2}, \qquad 0^\circ \le x < 180^\circ.$$

First solve for the angle

$$2x.$$

Since

$$\sin\theta = \frac{\sqrt{3}}{2},$$

we have

$$\theta = 60^\circ,\ 120^\circ,$$

together with coterminal angles differing by

$$360^\circ.$$

Because

$$0^\circ \le x < 180^\circ,$$

it follows that

$$0^\circ \le 2x < 360^\circ.$$

Hence,

$$2x = 60^\circ$$

or

$$2x = 120^\circ.$$

Dividing by 2 gives

$$\boxed{x = 30^\circ,\ 60^\circ.}$$

Practice Problems

1. Solve each triangle using the Law of Sines.
   (a) $\alpha = 38^\circ,\ \beta = 71^\circ,\ a = 12$
   (b) $\alpha = 52^\circ,\ a = 15,\ b = 18$

2. Solve each triangle using the Law of Cosines.
   (a) $a = 8,\ b = 11,\ \gamma = 47^\circ$
   (b) $a = 7,\ b = 9,\ c = 12$

3. Simplify each expression.
   (a) $\sec x \tan x$
   (b) $\dfrac{1-\cos^2 x}{\sin x}$
   (c) $\dfrac{\tan x}{\sec x}$

4. Evaluate exactly.
   (a) $\sin 75^\circ$
   (b) $\cos 15^\circ$
   (c) $\tan 75^\circ$

5. Use a double-angle formula to evaluate.
   (a) $\sin 60^\circ$
   (b) $\cos 120^\circ$
   (c) $\tan 90^\circ$ (if defined)

6. Evaluate using a half-angle formula.
   (a) $\sin 75^\circ$
   (b) $\cos 75^\circ$

7. Solve on $0^\circ \le x < 360^\circ$.
   (a) $\sin x = \dfrac{1}{2}$
   (b) $\cos x = -\dfrac{\sqrt{2}}{2}$
   (c) $\tan x = -1$
   (d) $2\sin x - 1 = 0$
   (e) $2\cos^2 x - 1 = 0$