Calculus II

Lecture 1

Contents

1. Integration by Parts

Integration by parts is one of the fundamental techniques of integration. Unlike substitution, which simplifies an integral by changing variables, integration by parts is designed for integrals involving the product of two functions.
Typical examples include

$$\int x \sin x \, dx, \qquad \int x e^x \, dx, \qquad \int x^2 e^x \, dx, \qquad \int \ln x \, dx.$$

In each of these examples the integrand is naturally viewed as the product of two functions. The strategy is to differentiate one factor while integrating the other. If the functions are chosen wisely, the resulting integral is simpler than the original one.
The integration by parts formula is not an isolated result. It follows directly from one of the most important differentiation rules studied in Calculus I: the Product Rule.

1.1 Review of the Product Rule

Recall that if

$$F(x) = f(x)g(x),$$

then

$$\boxed{\frac{d}{dx}\left(f(x)g(x)\right) = f'(x)g(x) + f(x)g'(x).}$$

This formula tells us how to differentiate the product of two functions.
Our goal is to reverse this formula to obtain an integration formula.

1.2 Derivation of the Integration by Parts Formula

Starting with the Product Rule,

$$\frac{d}{dx}\left(f(x)g(x)\right) = f'(x)g(x) + f(x)g'(x),$$

integrate both sides with respect to x.

$$\int \frac{d}{dx}\left(f(x)g(x)\right) \, dx = \int \left(f'(x)g(x) + f(x)g'(x)\right) dx.$$

Since integration and differentiation are inverse operations,

$$f(x)g(x) = \int f'(x)g(x) \, dx + \int f(x)g'(x) \, dx.$$

Now solve for the second integral.
Subtract

$$\int f'(x)g(x) \, dx$$

from both sides.
This gives

$$\boxed{\int f(x)g'(x) \, dx = f(x)g(x) - \int f'(x)g(x) \, dx.}$$

This is the integration by parts formula written using the functions $f(x)$ and $g(x)$.

1.3 The Standard Notation

Instead of writing

$$f(x) \quad \text{and} \quad g(x),$$

we introduce the notation

$$u = f(x), \qquad dv = g'(x)\,dx.$$

Differentiating and integrating give

$$du = f'(x)\,dx,$$

and

$$v = \int dv.$$

Substituting these expressions into the previous formula yields the familiar integration by parts formula.

$$\boxed{\int u\,dv = uv - \int v\,du.}$$

This is the form used throughout Calculus II.

1.4 Interpretation

The formula

$$\int u\,dv = uv - \int v\,du$$

shows that we exchange differentiation and integration.
Initially,

$$u$$

is differentiated,
while

$$dv$$

is integrated.
The hope is that the new integral

$$\int v\,du$$

is easier than the original one.
The success of the method therefore depends on making a good choice of u and dv.

1.5 Choosing u and dv

Although there is no universal rule, experience suggests choosing

- u to be the function that becomes simpler after differentiation,
- dv to be the function that is easy to integrate.

A commonly used guideline is the **LIATE Rule**.

$$\boxed{L > I > A > T > E}$$

where

L	Logarithmic functions
I	Inverse trigonometric functions
A	Algebraic functions
T	Trigonometric functions
E	Exponential functions

When choosing u, select the function that appears earliest in this list.
This rule is only a guideline; it is not a theorem. Some integrals require a different choice in order to simplify the computation.

2. Definite Integrals

The integration by parts formula also applies to definite integrals.
Suppose we wish to evaluate

$$\int_a^b u\,dv.$$

Starting from the indefinite formula,

$$\int u\,dv = uv - \int v\,du,$$

evaluate both sides at the endpoints.
The result is

$$\boxed{\int_a^b u\,dv = uv\big|_a^b - \int_a^b v\,du.}$$

Here,

$$uv\big|_a^b = u(b)v(b) - u(a)v(a).$$

Notice that the boundary term must always be evaluated before computing the remaining integral.

Remarks The definite integral version differs from the indefinite version in only one way: the appearance of the boundary term

$$uv\big|_a^b.$$

Students often forget to evaluate this term after performing integration by parts. This is one of the most common mistakes when solving definite integral problems.
Throughout this chapter we will use both versions of the integration by parts formula depending on whether the integral has limits of integration.

3. Examples of Integration by Parts

The best way to learn integration by parts is through examples. In each example we shall follow the same general procedure.

1. Identify the two factors in the integrand.
2. Choose u and dv.

3. Compute du and v.
4. Apply the integration by parts formula.
5. Simplify the resulting integral.

The most important step is selecting appropriate choices for u and dv.

Example 1 Evaluate

$$I = \int x \sin x \, dx.$$

Step 1. Choose u and dv
The integrand is the product of

$$x \qquad \text{and} \qquad \sin x.$$

According to the LIATE guideline, algebraic functions are usually chosen as u.
Thus,

$$u = x, \qquad dv = \sin x \, dx.$$

Step 2. Compute du and v
Differentiate u.

$$du = dx.$$

Integrate dv.
Since

$$\int \sin x \, dx = -\cos x,$$

we have

$$v = -\cos x.$$

Step 3. Apply Integration by Parts
Recall the formula

$$\int u \, dv = uv - \int v \, du.$$

Substituting our choices,

$$I = x(-\cos x) - \int (-\cos x) \, dx.$$

Simplify.

$$I = -x \cos x + \int \cos x \, dx.$$

Since

$$\int \cos x \, dx = \sin x,$$

we obtain

$$\boxed{\int x \sin x \, dx = -x \cos x + \sin x + C.}$$

Verification
Differentiate the answer.

$$\frac{d}{dx}\left(-x \cos x + \sin x\right).$$

Using the Product Rule,

$$\begin{aligned}\frac{d}{dx}(-x \cos x) &= -\cos x + x \sin x, \\ \frac{d}{dx}(\sin x) &= \cos x.\end{aligned}$$

Adding these derivatives,

$$-\cos x + x \sin x + \cos x = x \sin x.$$

Therefore,

$$\boxed{\frac{d}{dx}\left(-x \cos x + \sin x\right) = x \sin x,}$$

confirming that our antiderivative is correct.

Example 2 Evaluate

$$I = \int x e^x \, dx.$$

Again, the integrand is the product of an algebraic function and an exponential function. Choose

$$u = x, \qquad dv = e^x \, dx.$$

Then

$$du = dx, \qquad v = e^x.$$

Applying integration by parts,

$$\begin{aligned} I &= x e^x - \int e^x \, dx \\ &= x e^x - e^x + C. \end{aligned}$$

Factor the exponential.

$$\boxed{\int x e^x \, dx = e^x (x - 1) + C.}$$

Example 3 Evaluate

$$I = \int x \cos x \, dx.$$

Choose

$$u = x, \qquad dv = \cos x \, dx.$$

Then

$$du = dx, \qquad v = \sin x.$$

Using integration by parts,

$$\begin{aligned} I &= x \sin x - \int \sin x \, dx \\ &= x \sin x + \cos x + C. \end{aligned}$$

Therefore,

$$\boxed{\int x \cos x \, dx = x \sin x + \cos x + C.}$$

Remarks These three examples illustrate an important pattern.
Whenever the integrand contains

$$x \sin x, \qquad x \cos x, \qquad xe^x,$$

the algebraic function

$$x$$

is differentiated, while the trigonometric or exponential function is integrated.
Differentiating a polynomial reduces its degree, making the resulting integral simpler than the original one. This principle extends naturally to higher degree polynomials, as we shall see in the next example involving $\int x^2 e^x \, dx$.

Example 4: Repeated Integration by Parts Evaluate

$$I = \int x^2 e^x \, dx.$$

Unlike the previous examples, a single application of integration by parts does not completely evaluate the integral. Instead, the technique must be applied repeatedly until the remaining integral can be computed directly.

Step 1. Choose u and dv
The integrand is the product of

$$x^2 \qquad \text{and} \qquad e^x.$$

According to the LIATE guideline, choose

$$u = x^2, \qquad dv = e^x \, dx.$$

Differentiate and integrate.

$$du = 2x\,dx,$$

$$v = e^x.$$

Step 2. Apply Integration by Parts
Using

$$\int u\,dv = uv - \int v\,du,$$

we obtain

$$\begin{aligned} I &= x^2e^x - \int 2xe^x\,dx \\ &= x^2e^x - 2\int xe^x\,dx. \end{aligned}$$

The remaining integral

$$\int xe^x\,dx$$

is still not elementary, but it is simpler than the original integral because the degree of the polynomial has decreased.

Step 3. Evaluate the Remaining Integral
From Example 2,

$$\int xe^x\,dx = xe^x - e^x + C.$$

Substitute this result.

$$\begin{aligned} I &= x^2e^x - 2(xe^x - e^x) + C \\ &= x^2e^x - 2xe^x + 2e^x + C. \end{aligned}$$

Factor the exponential function.

$$\boxed{\int x^2e^x\,dx = e^x(x^2 - 2x + 2) + C.}$$

Verification
Differentiate

$$e^x(x^2 - 2x + 2).$$

Using the Product Rule,

$$\begin{aligned} \frac{d}{dx}\left[e^x(x^2 - 2x + 2)\right] &= e^x(x^2 - 2x + 2) + e^x(2x - 2) \\ &= e^x(x^2). \end{aligned}$$

Hence,

$$\boxed{\frac{d}{dx}\left[e^x(x^2-2x+2)\right]=x^2e^x,}$$

confirming the answer.

Example 5: Integrating a Logarithmic Function Evaluate

$$I=\int \ln x\,dx.$$

At first glance this integral appears unsuitable for integration by parts, since only one function is visible.
The key observation is that

$$\ln x = 1\cdot \ln x.$$

Thus the integrand is still the product of two functions.

Step 1. Choose u and dv
Choose

$$u=\ln x,\qquad dv=dx.$$

Differentiate and integrate.

$$du=\frac{1}{x}\,dx,$$

$$v=x.$$

Step 2. Apply Integration by Parts
Using the formula,

$$\begin{aligned}I&=x\ln x-\int x\left(\frac{1}{x}\right)dx\\&=x\ln x-\int 1\,dx.\end{aligned}$$

Since

$$\int 1\,dx=x,$$

we obtain

$$\boxed{\int \ln x\,dx = x\ln x - x + C.}$$

Verification
Differentiate

$$x\ln x - x.$$

Using the Product Rule,

$$\frac{d}{dx}(x \ln x) = \ln x + 1,$$

and

$$\frac{d}{dx}(-x) = -1.$$

Therefore,

$$(\ln x + 1) - 1 = \ln x.$$

Hence,

$$\boxed{\frac{d}{dx}(x \ln x - x) = \ln x.}$$

3.1 Discussion

Examples 4 and 5 illustrate two important ideas.

1. Sometimes integration by parts must be applied more than once. Each application reduces the degree of the polynomial until the remaining integral can be evaluated directly.
2. A function that appears to stand alone, such as

$$\ln x,$$

can always be rewritten as

$$1 \cdot \ln x,$$

allowing integration by parts to be applied.

These techniques are used frequently in later chapters, particularly when integrating products of algebraic, exponential, logarithmic, and inverse trigonometric functions.

4. Review of Trigonometric Identities

Before studying trigonometric integrals, we review the identities that will be used throughout this chapter. Although these formulas were introduced in Calculus I and earlier trigonometry courses, they play a central role in simplifying integrals involving powers and products of trigonometric functions. Many trigonometric integrals cannot be evaluated until one or more of these identities is applied.

4.1 Pythagorean Identities

The most fundamental trigonometric identities are the Pythagorean identities.

$$\boxed{\sin^2 \theta + \cos^2 \theta = 1}$$

$$\boxed{1 + \tan^2 \theta = \sec^2 \theta}$$

These identities are valid for every real value of θ.

Equivalent Forms

Depending on the problem, it is often convenient to solve these identities for one of the squared functions.

From

$$\sin^2\theta + \cos^2\theta = 1,$$

we obtain

$$\boxed{\sin^2\theta = 1 - \cos^2\theta,}$$

and

$$\boxed{\cos^2\theta = 1 - \sin^2\theta.}$$

Similarly,

$$1 + \tan^2\theta = \sec^2\theta$$

can be rewritten as

$$\boxed{\tan^2\theta = \sec^2\theta - 1.}$$

These alternate forms are used constantly when evaluating trigonometric integrals.

4.2 Why These Identities Matter

Suppose we wish to evaluate

$$\int \sin^3 x\,dx.$$

Since

$$\sin^3 x = \sin x \sin^2 x,$$

we may write

$$\sin^2 x = 1 - \cos^2 x.$$

Therefore,

$$\sin^3 x = \sin x(1 - \cos^2 x),$$

which can now be integrated using the substitution

$$u = \cos x.$$

Without the identity

$$\sin^2 x = 1 - \cos^2 x,$$

this substitution would not be possible.

4.3 Double-Angle Identities

The next identities involve angles of the form 2θ.
They allow products of sine and cosine to be rewritten as single trigonometric functions.
The double-angle identities are

$$\boxed{\sin(2\theta) = 2\sin\theta\cos\theta.}$$

and

$$\boxed{\cos(2\theta) = \cos^2\theta - \sin^2\theta.}$$

Since

$$\sin^2\theta + \cos^2\theta = 1,$$

the cosine double-angle formula can also be written in two additional forms.
Replacing

$$\sin^2\theta = 1 - \cos^2\theta,$$

gives

$$\boxed{\cos(2\theta) = 2\cos^2\theta - 1.}$$

Replacing

$$\cos^2\theta = 1 - \sin^2\theta,$$

gives

$$\boxed{\cos(2\theta) = 1 - 2\sin^2\theta.}$$

All three formulas are equivalent.

4.4 Choosing the Appropriate Form

The appropriate version depends on the problem.

- If the integral contains powers of cosine, use

$$\cos(2\theta) = 2\cos^2\theta - 1.$$

- If the integral contains powers of sine, use

$$\cos(2\theta) = 1 - 2\sin^2\theta.$$

- If both sine and cosine appear, use

$$\cos(2\theta) = \cos^2\theta - \sin^2\theta.$$

4.5 Half-Angle (Lowering Power) Identities

One of the most important applications of the double-angle identities is the derivation of the half-angle formulas.
These formulas reduce even powers of sine and cosine to first powers of cosine.
Starting with

$$\cos(2\theta) = 1 - 2\sin^2\theta,$$

solve for

$$\sin^2\theta.$$

We obtain

$$2\sin^2\theta = 1 - \cos(2\theta),$$

so

$$\boxed{\sin^2\theta = \frac{1-\cos(2\theta)}{2}.}$$

Similarly,
from

$$\cos(2\theta) = 2\cos^2\theta - 1,$$

we obtain

$$2\cos^2\theta = 1 + \cos(2\theta),$$

and therefore

$$\boxed{\cos^2\theta = \frac{1+\cos(2\theta)}{2}.}$$

4.6 Applications of the Half-Angle Formulas

These identities are especially useful when both sine and cosine have even powers.
For example,

$$\int \cos^2 x\, dx$$

cannot be evaluated directly.
Instead,

$$\cos^2 x = \frac{1+\cos 2x}{2},$$

so

$$\begin{aligned}\int \cos^2 x\, dx &= \frac{1}{2}\int (1+\cos 2x)\, dx\\ &= \frac{x}{2} + \frac{\sin 2x}{4} + C.\end{aligned}$$

The lowering power identities convert an integral involving a square into one that can be integrated using elementary techniques.

4.7 Summary of Trigonometric Identities

For convenience, we summarize the identities introduced in this section.

$$\boxed{\begin{aligned} \sin^2\theta+\cos^2\theta&=1,\\ 1+\tan^2\theta&=\sec^2\theta,\\ \sin(2\theta)&=2\sin\theta\cos\theta,\\ \cos(2\theta)&=\cos^2\theta-\sin^2\theta,\\ \sin^2\theta&=\frac{1-\cos2\theta}{2},\\ \cos^2\theta&=\frac{1+\cos2\theta}{2}.\end{aligned}}$$

These identities form the foundation for every strategy discussed in the next section on trigonometric integrals.

5. Review of Trigonometric Derivatives and Integrals

Before developing techniques for evaluating trigonometric integrals, it is helpful to review the basic derivatives and antiderivatives of the six trigonometric functions.

Many of the examples in this chapter require these formulas repeatedly. A common source of mistakes is forgetting the sign or confusing the derivatives of the reciprocal trigonometric functions.

5.1 Derivatives of the Trigonometric Functions

The six basic derivative formulas are summarized below.

$$\boxed{\begin{aligned} \frac{d}{dx}(\sin x)&=\cos x,\\ \frac{d}{dx}(\cos x)&=-\sin x,\\ \frac{d}{dx}(\tan x)&=\sec^2 x,\\ \frac{d}{dx}(\cot x)&=-\csc^2 x,\\ \frac{d}{dx}(\sec x)&=\sec x\tan x,\\ \frac{d}{dx}(\csc x)&=-\csc x\cot x.\end{aligned}}$$

These formulas should be memorized.

5.2 Observations

Notice the following patterns.

- The derivatives of sine and cosine differ only by a negative sign.
- The derivatives of tangent and secant are positive.
- The derivatives of cotangent and cosecant are negative.
- Whenever secant is differentiated, tangent also appears.
- Whenever cosecant is differentiated, cotangent also appears.

Recognizing these patterns makes the formulas easier to remember.

5.3 Standard Trigonometric Integrals

Since differentiation and integration are inverse operations, every derivative formula immediately produces an integration formula.

The most frequently used antiderivatives are

$$\boxed{\begin{aligned}\int \cos x\,dx &= \sin x + C,\\ \int \sin x\,dx &= -\cos x + C,\\ \int \sec^2 x\,dx &= \tan x + C,\\ \int \csc^2 x\,dx &= -\cot x + C,\\ \int \sec x \tan x\,dx &= \sec x + C,\\ \int \csc x \cot x\,dx &= -\csc x + C.\end{aligned}}$$

These six integrals appear repeatedly throughout this chapter.

5.4 Verification

As an illustration, consider

$$\int \sec x \tan x\,dx.$$

Since

$$\frac{d}{dx}(\sec x) = \sec x \tan x,$$

it follows immediately that

$$\boxed{\int \sec x \tan x\,dx = \sec x + C.}$$

Similarly,

$$\frac{d}{dx}(-\csc x) = \csc x \cot x,$$

which implies

$$\boxed{\int \csc x \cot x\,dx = -\csc x + C.}$$

5.5 Integrals That Are *Not* Elementary

Not every trigonometric integral can be evaluated directly from the previous table.

For example,

$$\int \sin^2 x\,dx, \qquad \int \cos^4 x\,dx, \qquad \int \sin^3 x \cos^2 x\,dx$$

cannot be computed using only the formulas above.
Instead, we first rewrite the integrand using trigonometric identities.
This observation motivates the techniques developed in the next section.

Example Evaluate

$$\int \sec x \tan x \, dx.$$

Because

$$\frac{d}{dx}(\sec x) = \sec x \tan x,$$

we immediately obtain

$$\boxed{\int \sec x \tan x \, dx = \sec x + C.}$$

Next consider

$$\int \sin^2 x \, dx.$$

No formula from the previous table applies directly.
Using the lowering power identity,

$$\sin^2 x = \frac{1 - \cos 2x}{2},$$

we obtain

$$\begin{aligned} \int \sin^2 x \, dx &= \frac{1}{2} \int (1 - \cos 2x) \, dx \\ &= \frac{x}{2} - \frac{\sin 2x}{4} + C. \end{aligned}$$

This example illustrates an important principle.

> Whenever powers of sine or cosine appear, begin by asking whether a trigonometric identity should be applied before attempting to integrate.

5.6 Transition to Trigonometric Integrals

The derivative and integral formulas reviewed in this section provide the basic building blocks for evaluating trigonometric integrals.
The real challenge is determining *how to rewrite* complicated powers and products of trigonometric functions so that these elementary formulas become applicable.
The next section develops systematic strategies for evaluating integrals such as

$$\int \sin^5 x \cos^2 x \, dx,$$

$$\int \sin^4 x \cos^2 x \, dx,$$

and

$$\int \tan^3 x \sec^4 x \, dx.$$

Rather than memorizing individual examples, we shall develop general rules that apply to each class of trigonometric integrals.

6. Strategies for Products of Sine and Cosine

In the previous section we reviewed the trigonometric identities and standard integrals that will be used throughout this chapter. We now combine those ideas into a systematic procedure for evaluating integrals involving products of powers of sine and cosine.

Our goal is to evaluate integrals of the general form

$$\boxed{\int \sin^m x \cos^n x \, dx,}$$

where m and n are nonnegative integers.

Unlike substitution or integration by parts, there is no single technique that works for every choice of m and n. Instead, the appropriate method depends entirely on whether the exponents are odd or even. The following three cases cover every possible situation.

6.1 Case 1: The Power of Cosine is Odd

Suppose

$$n = 2k + 1,$$

where k is a nonnegative integer.

In other words, the exponent of cosine is odd.

Examples include

$$\cos x, \qquad \cos^3 x, \qquad \cos^5 x, \qquad \cos^7 x.$$

Strategy

When the cosine exponent is odd,

1. Save one factor of $\cos x$.
2. Rewrite the remaining cosine factors using

$$\boxed{\cos^2 x = 1 - \sin^2 x.}$$

3. Use the substitution

$$u = \sin x.$$

Why Does This Work?

Saving one cosine is not an arbitrary trick.

The derivative of

$$\sin x$$

is

$$\cos x.$$

Therefore,

$$du = \cos x \, dx,$$

which appears naturally after saving one cosine factor.
The remaining cosine factors can then be converted into powers of sine using

$$\cos^2 x = 1 - \sin^2 x.$$

The resulting integral becomes an ordinary polynomial integral.

Example 1 Evaluate

$$\int \sin^2 x \cos^3 x \, dx.$$

Separate one cosine factor.

$$\cos^3 x = \cos^2 x \cos x.$$

Therefore,

$$\int \sin^2 x \cos^3 x \, dx = \int \sin^2 x \cos^2 x \cos x \, dx.$$

Replace

$$\cos^2 x = 1 - \sin^2 x.$$

Then

$$I = \int \sin^2 x (1 - \sin^2 x) \cos x \, dx.$$

Now let

$$u = \sin x.$$

Then

$$du = \cos x \, dx.$$

Substituting,

$$\begin{aligned} I &= \int u^2 (1 - u^2) \, du \\ &= \int (u^2 - u^4) \, du \\ &= \frac{u^3}{3} - \frac{u^5}{5} + C. \end{aligned}$$

Finally,

$$\boxed{\int \sin^2 x \cos^3 x \, dx = \frac{\sin^3 x}{3} - \frac{\sin^5 x}{5} + C.}$$

6.2 Case 2: The Power of Sine is Odd

Suppose

$$m = 2k + 1.$$

The exponent of sine is odd.
Examples include

$$\sin x, \qquad \sin^3 x, \qquad \sin^5 x.$$

Strategy

1. Save one factor of $\sin x$.
2. Rewrite the remaining sine factors using

$$\boxed{\sin^2 x = 1 - \cos^2 x.}$$

3. Use

$$u = \cos x.$$

Why Does This Work?
Since

$$\frac{d}{dx}(\cos x) = -\sin x,$$

saving one sine provides the differential needed for substitution.
The negative sign is absorbed into the substitution.

Example 2 Evaluate

$$\int \sin^3 x \cos^4 x \, dx.$$

Write

$$\sin^3 x = \sin^2 x \sin x.$$

Replace

$$\sin^2 x = 1 - \cos^2 x.$$

Then

$$I = \int (1 - \cos^2 x) \cos^4 x \sin x \, dx.$$

Let

$$u = \cos x,$$

so

$$du = -\sin x \, dx.$$

Hence,

$$\begin{aligned} I &= -\int (1-u^2)u^4\,du \\ &= -\int (u^4-u^6)\,du \\ &= -\frac{u^5}{5}+\frac{u^7}{7}+C. \end{aligned}$$

Therefore,

$$\boxed{\int \sin^3 x\cos^4 x\,dx = -\frac{\cos^5 x}{5}+\frac{\cos^7 x}{7}+C.}$$

6.3 Case 3: Both Powers are Even

The remaining possibility is that

$$m$$

and

$$n$$

are both even.
Typical examples include

$$\sin^2 x\cos^2 x,$$

$$\sin^4 x,$$

$$\cos^6 x.$$

In this case there is no odd sine or cosine factor available for substitution.
Instead we use the lowering power identities

$$\boxed{\sin^2 x = \frac{1-\cos 2x}{2},}$$

and

$$\boxed{\cos^2 x = \frac{1+\cos 2x}{2}.}$$

These identities reduce the powers until the resulting integral can be evaluated directly.

6.4 Decision Tree

Whenever you encounter

$$\int \sin^m x\cos^n x\,dx,$$

use the following procedure.

Is the cosine exponent odd?
$\Downarrow$
Yes $\rightarrow$ Save one cosine and use $u = \sin x$.

Otherwise, is the sine exponent odd?
$\Downarrow$
Yes $\rightarrow$ Save one sine and use $u = \cos x$.

Otherwise
$\Downarrow$
Use half-angle identities.

Example 3: Both Powers are Even Evaluate

$$I = \int \sin^2 x \, dx.$$

At first glance, none of the standard integration formulas applies.
Although we know

$$\int \sin x \, dx = -\cos x,$$

there is no elementary formula for

$$\int \sin^2 x \, dx.$$

Since the exponent of sine is even, the appropriate strategy is to use the lowering power identity.

Step 1. Apply the Half-Angle Identity
Recall

$$\boxed{\sin^2 x = \frac{1 - \cos 2x}{2}.}$$

Substitute this identity into the integral.

$$\begin{aligned} I &= \int \frac{1 - \cos 2x}{2} \, dx \\ &= \frac{1}{2} \int (1 - \cos 2x) \, dx. \end{aligned}$$

Step 2. Integrate
Distribute the integral.

$$I = \frac{1}{2} \int 1 \, dx - \frac{1}{2} \int \cos 2x \, dx.$$

The first integral is immediate.

$$\int 1 \, dx = x.$$

For the second integral, let

$$u = 2x.$$

Then

$$du = 2dx, \qquad dx = \frac{1}{2}du.$$

Therefore,

$$\begin{aligned}\int \cos 2x\, dx &= \frac{1}{2}\int \cos u\, du \\ &= \frac{1}{2}\sin u \\ &= \frac{1}{2}\sin 2x.\end{aligned}$$

Substituting,

$$\begin{aligned}I &= \frac{1}{2}x - \frac{1}{2}\left(\frac{1}{2}\sin 2x\right) + C \\ &= \boxed{\frac{x}{2} - \frac{\sin 2x}{4} + C.}\end{aligned}$$

Verification
Differentiate the answer.

$$\frac{d}{dx}\left(\frac{x}{2} - \frac{\sin 2x}{4}\right).$$

Using the Chain Rule,

$$= \frac{1}{2} - \frac{1}{4}(2\cos 2x) = \frac{1}{2} - \frac{1}{2}\cos 2x.$$

Using the identity

$$\sin^2 x = \frac{1 - \cos 2x}{2},$$

we recover

$$\boxed{\frac{1}{2} - \frac{1}{2}\cos 2x = \sin^2 x.}$$

Thus the antiderivative is correct.

Example 4 Evaluate

$$I = \int \cos^2 x\, dx.$$

Again, the exponent is even, so we use the lowering power identity.

Step 1. Rewrite the Integrand
Recall

$$\boxed{\cos^2 x = \frac{1+\cos 2x}{2}.}$$

Therefore,

$$\begin{aligned} I &= \int \frac{1+\cos 2x}{2} dx \\ &= \frac{1}{2}\int (1+\cos 2x) dx. \end{aligned}$$

Step 2. Integrate
Proceed exactly as before.

$$\begin{aligned} I &= \frac{1}{2}x + \frac{1}{2}\left(\frac{1}{2}\sin 2x\right) + C \\ &= \boxed{\frac{x}{2} + \frac{\sin 2x}{4} + C.} \end{aligned}$$

Example 5 Evaluate

$$I = \int \sin^2 x \cos^2 x \, dx.$$

Both exponents are even.
Therefore the half-angle identities must be used.

Step 1. Rewrite One Factor
Use

$$\sin^2 x = \frac{1-\cos 2x}{2},$$

and

$$\cos^2 x = \frac{1+\cos 2x}{2}.$$

Then

$$\begin{aligned} \sin^2 x \cos^2 x &= \frac{1}{4}(1-\cos 2x)(1+\cos 2x) \\ &= \frac{1}{4}(1-\cos^2 2x). \end{aligned}$$

Notice that another even power remains.
Apply the identity again.

$$\cos^2 2x = \frac{1+\cos 4x}{2}.$$

Hence,

$$\begin{aligned} 1-\cos^2 2x &= 1 - \frac{1+\cos 4x}{2} \\ &= \frac{1-\cos 4x}{2}. \end{aligned}$$

Therefore,

$$\boxed{\sin^2 x \cos^2 x = \frac{1}{8}(1 - \cos 4x).}$$

Step 2. Integrate
Now the integral is straightforward.

$$\begin{aligned} I &= \frac{1}{8}\int (1 - \cos 4x)dx \\ &= \frac{x}{8} - \frac{1}{8}\left(\frac{1}{4}\sin 4x\right) + C. \end{aligned}$$

Thus,

$$\boxed{\int \sin^2 x \cos^2 x \, dx = \frac{x}{8} - \frac{\sin 4x}{32} + C.}$$

6.5 Summary of Cases

The three cases for

$$\int \sin^m x \cos^n x \, dx$$

are summarized below.

Situation	**Method**
Odd cosine power	Save one cosine, $u = \sin x$
Odd sine power	Save one sine, $u = \cos x$
Both even	Use half-angle identities

These three rules completely determine the method for evaluating every integral involving only powers of sine and cosine.

7. Strategies for Powers of Tangent and Secant

The second major class of trigonometric integrals involves powers of tangent and secant.
Our objective is to evaluate integrals of the form

$$\boxed{\int \tan^m x \sec^n x \, dx,}$$

where m and n are nonnegative integers.
Unlike the sine-cosine case, the strategy depends upon whether the power of secant is even or whether the power of tangent is odd.
These strategies arise from the two identities

$$\boxed{1 + \tan^2 x = \sec^2 x,}$$

and

$$\boxed{\frac{d}{dx}(\tan x) = \sec^2 x, \qquad \frac{d}{dx}(\sec x) = \sec x \tan x.}$$

Notice that each derivative naturally suggests an appropriate substitution.

7.1 Case 1: The Power of Secant is Even

Suppose the exponent of secant is an even positive integer.
Typical examples include

$$\sec^2 x, \qquad \sec^4 x, \qquad \sec^6 x.$$

Strategy
Proceed as follows.

1. Save one factor of
$$\sec^2 x.$$
2. Rewrite the remaining secant factors using
$$\boxed{\sec^2 x = 1 + \tan^2 x.}$$
3. Use the substitution
$$u = \tan x.$$
Since
$$du = \sec^2 x\, dx,$$
the substitution follows naturally.

Example 1 Evaluate

$$I = \int \tan^3 x \sec^4 x\, dx.$$

Separate one factor of

$$\sec^2 x.$$

$$I = \int \tan^3 x \sec^2 x \sec^2 x\, dx.$$

Use

$$\sec^2 x = 1 + \tan^2 x.$$

Then

$$I = \int \tan^3 x (1 + \tan^2 x) \sec^2 x dx.$$

Now let

$$u = \tan x.$$

Then

$$du = \sec^2 x\,dx.$$

Therefore,

$$\begin{aligned} I &= \int u^3(1+u^2)\,du \\ &= \int (u^3+u^5)\,du \\ &= \frac{u^4}{4} + \frac{u^6}{6} + C. \end{aligned}$$

Hence,

$$\boxed{\int \tan^3 x \sec^4 x\,dx = \frac{\tan^4 x}{4} + \frac{\tan^6 x}{6} + C.}$$

7.2 Case 2: The Power of Tangent is Odd

Suppose the exponent of tangent is odd.
Examples include

$$\tan x, \qquad \tan^3 x, \qquad \tan^5 x.$$

Strategy
When the tangent exponent is odd,

1. Save one factor

$$\sec x \tan x.$$

2. Rewrite the remaining tangent factors using

$$\boxed{\tan^2 x = \sec^2 x - 1.}$$

3. Use

$$u = \sec x.$$

Since

$$du = \sec x \tan x\,dx,$$

the substitution is immediate.

Example 2 Evaluate

$$I = \int \tan^3 x \sec x\,dx.$$

Write

$$\tan^3 x = \tan^2 x \tan x.$$

Therefore,

$$I = \int (\tan^2 x)(\sec x \tan x) dx.$$

Replace

$$\tan^2 x = \sec^2 x - 1.$$

Then

$$I = \int (\sec^2 x - 1)(\sec x \tan x) dx.$$

Let

$$u = \sec x.$$

Then

$$du = \sec x \tan x \, dx.$$

Hence,

$$\begin{aligned} I &= \int (u^2 - 1) du \\ &= \frac{u^3}{3} - u + C. \end{aligned}$$

Substituting,

$$\boxed{\int \tan^3 x \sec x \, dx = \frac{\sec^3 x}{3} - \sec x + C.}$$

7.3 Decision Tree

For integrals involving tangent and secant, use the following guide.

Is the secant exponent even?

$\Downarrow$

Yes $\to$ Save $\sec^2 x, \; u = \tan x.$

Otherwise, is the tangent exponent odd?

$\Downarrow$

Yes $\to$ Save $\sec x \tan x, \; u = \sec x.$

Otherwise

$\Downarrow$

Rewrite the integral using identities.

7.4 Comparison of the Two Main Strategies

By this point we have developed two major techniques for trigonometric integrals.

Integral Type	Primary Strategy
$\int \sin^m x \cos^n x \, dx$	Odd cosine: $u = \sin x$ Odd sine: $u = \cos x$ Both even: Half-angle identities
$\int \tan^m x \sec^n x \, dx$	Even secant: $u = \tan x$ Odd tangent: $u = \sec x$

These decision rules cover the majority of trigonometric integrals encountered in Calculus II. Before beginning any computation, first identify the form of the integrand and determine which strategy applies.

7.5 Common Mistakes

Students frequently make the following errors.

1. Using the identity

$$\tan^2 x = \sec^2 x - 1$$

   when

$$\sec^2 x = 1 + \tan^2 x$$

   would simplify the computation more naturally.

2. Forgetting to save one factor of

$$\sec^2 x$$

   or

$$\sec x \tan x.$$

3. Choosing the wrong substitution.
4. Attempting substitution before rewriting the integrand.
5. Forgetting that the strategy depends on the parity (odd or even) of the exponents.

8. Combining Integration Techniques

In many applications, a single integration technique is not sufficient.
Instead, several techniques must be combined in a logical order.
Typical combinations include

- algebraic simplification followed by substitution,
- trigonometric identities followed by substitution,
- substitution followed by integration by parts,
- repeated applications of integration by parts.

Before attempting a difficult integral, always ask whether the integrand can be simplified first.

Example Evaluate

$$I = \int_0^{\pi} e^{\cos t} \sin(2t)\, dt.$$

Step 1. Simplify the Integrand
The expression

$$\sin(2t)$$

should immediately suggest the double-angle identity

$$\boxed{\sin(2t) = 2\sin t \cos t.}$$

Substituting,

$$\begin{aligned} I &= \int_0^{\pi} e^{\cos t}(2\sin t \cos t) dt \\ &= 2\int_0^{\pi} e^{\cos t} \sin t \cos t dt. \end{aligned}$$

The integral now contains the factor

$$\sin t\, dt,$$

which is the derivative (up to sign) of

$$\cos t.$$

Step 2. Make a Substitution
Let

$$u = \cos t.$$

Then

$$du = -\sin t\, dt.$$

Also change the limits of integration.
When

$$t = 0,$$

$$u = \cos 0 = 1.$$

When

$$t = \pi,$$

$$u = \cos \pi = -1.$$

Thus,

$$\begin{aligned} I &= -2\int_{1}^{-1} ue^u\,du \\ &= 2\int_{-1}^{1} ue^u\,du. \end{aligned}$$

Notice that the original trigonometric integral has now become an exponential integral.

Step 3. Apply Integration by Parts

Evaluate

$$\int ue^u\,du.$$

Choose

$$u_1 = u, \qquad dv = e^u\,du.$$

Then

$$du_1 = du, \qquad v = e^u.$$

Applying integration by parts,

$$\begin{aligned} \int ue^u\,du &= ue^u - \int e^u\,du \\ &= ue^u - e^u + C \\ &= e^u(u-1) + C. \end{aligned}$$

Step 4. Evaluate the Definite Integral

Substituting,

$$\begin{aligned} I &= 2\left[e^u(u-1)\right]_{-1}^{1} \\ &= 2\Big[e(0) - e^{-1}(-2)\Big] \\ &= 2\left(\frac{2}{e}\right) \\ &= \boxed{\frac{4}{e}.} \end{aligned}$$