Calculus II

Lecture 2

Contents

1. Inverse Trigonometric Functions

This section collects the three inverse trigonometric functions needed for the lecture: inverse sine, inverse cosine, and inverse tangent. The organizing principle is the same in every case: the original trigonometric function must be restricted to a one-to-one interval before its inverse can be defined.

1.1. The Inverse Sine Function

1.1.1 Why We Need to Restrict the Sine Function

The sine function

$$y = \sin x$$

is not one-to-one on its full domain $(-\infty, \infty)$ because many different angles have the same sine value. For example,

$$\sin\left(\frac{\pi}{6}\right) = \frac{1}{2}, \qquad \sin\left(\frac{5\pi}{6}\right) = \frac{1}{2}.$$

Therefore, the sine function does not have an inverse on all of $\mathbb{R}$.

To define an inverse sine function, we restrict sine to the interval

$$\left[-\frac{\pi}{2}, \frac{\pi}{2}\right].$$

On this interval, sine is one-to-one and takes every value between -1 and 1.

$$\sin : \left[-\frac{\pi}{2}, \frac{\pi}{2}\right] \to [-1, 1].$$

Therefore, the inverse sine function is defined from $[-1, 1]$ back to

$$\left[-\frac{\pi}{2}, \frac{\pi}{2}\right].$$

1.1.2 Definition of the Inverse Sine Function

Definition 1.1. The inverse sine function, denoted by $\sin^{-1}(x)$ or $\arcsin(x)$, is defined by

$$\sin^{-1}(x) = y \quad \iff \quad \sin y = x,$$

where

$$-\frac{\pi}{2} \le y \le \frac{\pi}{2}.$$

Thus, $\sin^{-1}(x)$ gives the unique angle in the interval

$$\left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$$

whose sine is x.

1.1.3 Domain and Range

The domain of $\sin^{-1}(x)$ is

$$[-1, 1],$$

because sine values cannot be less than -1 or greater than 1.

The range of $\sin^{-1}(x)$ is

$$\left[-\frac{\pi}{2}, \frac{\pi}{2}\right].$$

So,

$$\boxed{\sin^{-1} : [-1, 1] \to \left[-\frac{\pi}{2}, \frac{\pi}{2}\right]}$$

1.1.4 Important Properties

For every $x \in [-1, 1]$,

$$\sin(\sin^{-1} x) = x.$$

However,

$$\sin^{-1}(\sin\theta) = \theta$$

is true only if

$$-\frac{\pi}{2} \leq \theta \leq \frac{\pi}{2}.$$

This distinction is very important.

Remark 1.2. The expression $\sin^{-1}(x)$ does not mean

$$\frac{1}{\sin x}.$$

It means the inverse sine function. The reciprocal of sine is written as

$$\csc x = \frac{1}{\sin x}.$$

1.1.5 Principal Values

The value of $\sin^{-1}(x)$ is called the principal value. It is the angle chosen from the restricted range

$$\left[-\frac{\pi}{2}, \frac{\pi}{2}\right].$$

For example,

$$\sin\left(\frac{\pi}{6}\right) = \frac{1}{2}$$

and also

$$\sin\left(\frac{5\pi}{6}\right) = \frac{1}{2}.$$

But

$$\sin^{-1}\left(\frac{1}{2}\right) = \frac{\pi}{6},$$

not $\frac{5\pi}{6}$, because $\frac{5\pi}{6}$ is not in the principal range.

1.1.6 Examples

Example 1.3. Compute

$$\sin(\sin^{-1}(0.734)).$$

Since $0.734 \in [-1, 1]$, we can use the identity

$$\sin(\sin^{-1} x) = x.$$

Therefore,

$$\boxed{\sin(\sin^{-1}(0.734)) = 0.734.}$$

Example 1.4. Compute
$$\sin^{-1}(\sin(-50°)).$$

Since
$$-50° \in [-90°, 90°],$$
the angle is already in the principal range of inverse sine.

Therefore,
$$\boxed{\sin^{-1}(\sin(-50°)) = -50°.}$$

Example 1.5. Compute
$$\sin^{-1}\left(\sin\left(\frac{2\pi}{3}\right)\right).$$

First note that
$$\frac{2\pi}{3}$$
is not in the principal range
$$\left[-\frac{\pi}{2}, \frac{\pi}{2}\right].$$

Now,
$$\sin\left(\frac{2\pi}{3}\right) = \frac{\sqrt{3}}{2}.$$

Therefore,
$$\sin^{-1}\left(\sin\left(\frac{2\pi}{3}\right)\right) = \sin^{-1}\left(\frac{\sqrt{3}}{2}\right).$$

The angle in the principal range whose sine is $\frac{\sqrt{3}}{2}$ is
$$\frac{\pi}{3}.$$

Thus,
$$\boxed{\sin^{-1}\left(\sin\left(\frac{2\pi}{3}\right)\right) = \frac{\pi}{3}.}$$

Example 1.6. Compute
$$\sin^{-1}\left(\frac{1}{2}\right).$$

We need the angle θ in
$$\left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$$
such that
$$\sin\theta = \frac{1}{2}.$$

That angle is
$$\theta = \frac{\pi}{6}.$$

Therefore,
$$\boxed{\sin^{-1}\left(\frac{1}{2}\right) = \frac{\pi}{6}.}$$

Example 1.7. Compute
$$\sin^{-1}\left(-\frac{\sqrt{2}}{2}\right).$$

We need the angle θ in
$$\left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$$
such that
$$\sin\theta = -\frac{\sqrt{2}}{2}.$$

That angle is
$$\theta = -\frac{\pi}{4}.$$

Therefore,
$$\boxed{\sin^{-1}\left(-\frac{\sqrt{2}}{2}\right) = -\frac{\pi}{4}.}$$

Example 1.8. Compute
$$\cos\left(\sin^{-1}\left(\frac{5}{13}\right)\right).$$

Let
$$\theta = \sin^{-1}\left(\frac{5}{13}\right).$$

Then
$$\sin\theta = \frac{5}{13}.$$

Since sine is opposite over hypotenuse, we draw a right triangle with
$$\text{opposite} = 5, \qquad \text{hypotenuse} = 13.$$

By the Pythagorean theorem, the adjacent side is
$$\sqrt{13^2 - 5^2} = \sqrt{169 - 25} = \sqrt{144} = 12.$$

Thus,
$$\cos\theta = \frac{\text{adjacent}}{\text{hypotenuse}} = \frac{12}{13}.$$

Therefore,
$$\boxed{\cos\left(\sin^{-1}\left(\frac{5}{13}\right)\right) = \frac{12}{13}.}$$

1.1.7 Common Mistakes

- Confusing $\sin^{-1} x$ with $\frac{1}{\sin x}$.
- Forgetting that $\sin^{-1} x$ only accepts inputs from $[-1, 1]$.
- Forgetting that outputs of $\sin^{-1} x$ must lie in
$$\left[-\frac{\pi}{2}, \frac{\pi}{2}\right].$$
- Assuming $\sin^{-1}(\sin\theta) = \theta$ for all θ.

1.1.8 Practice Problems

Compute each value.

1. $\sin^{-1}(0)$
2. $\sin^{-1}(1)$
3. $\sin^{-1}(-1)$
4. $\sin^{-1}\left(\frac{\sqrt{3}}{2}\right)$
5. $\sin^{-1}\left(-\frac{1}{2}\right)$
6. $\sin^{-1}(\sin(5\pi/6))$
7. $\cos\left(\sin^{-1}\left(\frac{3}{5}\right)\right)$

1.1.9 Answers

1. 0
2. $\frac{\pi}{2}$
3. $-\frac{\pi}{2}$
4. $\frac{\pi}{3}$
5. $-\frac{\pi}{6}$
6. $\frac{\pi}{6}$
7. $\frac{4}{5}$

1.2. The Inverse Cosine Function

1.2.1 Motivation

Like the sine function, the cosine function is not one-to-one over its entire domain. For example,

$$\cos\left(\frac{\pi}{3}\right) = \frac{1}{2}, \qquad \cos\left(\frac{5\pi}{3}\right) = \frac{1}{2}.$$

Since multiple angles have the same cosine value, the cosine function has no inverse unless its domain is restricted.

To obtain an inverse function, we restrict the cosine function to the interval

$$[0, \pi].$$

On this interval, the cosine function is continuous and strictly decreasing from 1 to -1, making it one-to-one.

Therefore,

$$\cos : [0, \pi] \longrightarrow [-1, 1]$$

is invertible.

1.2.2 Definition of the Inverse Cosine Function

Definition 1.9. The inverse cosine function, denoted by

$$\cos^{-1}(x)$$

or

$$\arccos(x),$$

is defined by

$$\boxed{\cos^{-1}(x) = y \iff \cos y = x,}$$

where

$$0 \le y \le \pi.$$

Thus, $\cos^{-1}(x)$ is the unique angle in

$$[0, \pi]$$

whose cosine equals x.

1.2.3 Domain and Range

Since cosine values always satisfy

$$-1 \le \cos\theta \le 1,$$

the domain of $\cos^{-1}(x)$ is

$$[-1, 1].$$

The range is

$$[0, \pi].$$

Hence,

$$\boxed{\cos^{-1} : [-1, 1] \to [0, \pi].}$$

1.2.4 Properties

For every

$$x \in [-1, 1],$$

we have

$$\boxed{\cos(\cos^{-1} x) = x.}$$

However,

$$\cos^{-1}(\cos\theta) = \theta$$

is true only when

$$0 \leq \theta \leq \pi.$$

This is one of the most common mistakes students make.

1.2.5 Principal Values

The output of $\cos^{-1}(x)$ is called its principal value.

Among all angles having cosine equal to x, we choose the one lying in

$$[0, \pi].$$

For example,

$$\cos\left(\frac{\pi}{3}\right) = \frac{1}{2},$$

and

$$\cos\left(\frac{5\pi}{3}\right) = \frac{1}{2}.$$

Since

$$\frac{\pi}{3} \in [0, \pi],$$

but

$$\frac{5\pi}{3} \notin [0, \pi],$$

we obtain

$$\boxed{\cos^{-1}\left(\frac{1}{2}\right) = \frac{\pi}{3}.}$$

1.2.6 Example 1

Evaluate

$$\cos(\cos^{-1}(-0.35)).$$

Since

$$-0.35 \in [-1, 1],$$

we simply apply the identity

$$\cos(\cos^{-1} x) = x.$$

Therefore,

$$\boxed{\cos(\cos^{-1}(-0.35)) = -0.35.}$$

1.2.7 Example 2

Evaluate

$$\cos^{-1}\left(\cos\frac{2\pi}{3}\right).$$

Since

$$\frac{2\pi}{3}$$

already belongs to the principal interval

$$[0,\pi],$$

the inverse cosine returns the same angle.
Therefore,

$$\boxed{\cos^{-1}\left(\cos\frac{2\pi}{3}\right)=\frac{2\pi}{3}.}$$

1.2.8 Example 3

Evaluate

$$\cos^{-1}(\cos(-50^\circ)).$$

First compute

$$\cos(-50^\circ)=\cos(50^\circ),$$

because cosine is an even function.
Now

$$50^\circ$$

lies inside

$$[0^\circ,180^\circ],$$

which is the principal range of inverse cosine.
Hence,

$$\boxed{\cos^{-1}(\cos(-50^\circ))=50^\circ.}$$

Notice carefully that the answer is **not**

$$-50^\circ.$$

1.2.9 Example 4

Evaluate

$$\cos^{-1}\left(\frac{1}{2}\right).$$

We seek the angle

$$\theta \in [0, \pi]$$

such that

$$\cos\theta = \frac{1}{2}.$$

From the unit circle,

$$\boxed{\theta = \frac{\pi}{3}.}$$

1.2.10 Example 5

Evaluate

$$\cos^{-1}\left(-\frac{\sqrt{2}}{2}\right).$$

The cosine is negative in Quadrants II and III.
Since inverse cosine only returns angles in

$$[0, \pi],$$

the required angle is

$$\boxed{\frac{3\pi}{4}.}$$

1.2.11 Example 6

Evaluate

$$\tan\left(\cos^{-1}\left(\frac{7}{10}\right)\right).$$

Let

$$\theta = \cos^{-1}\left(\frac{7}{10}\right).$$

Then

$$\cos\theta = \frac{7}{10}.$$

Construct a right triangle.
Adjacent side:

$$7$$

Hypotenuse:

$$10.$$

The opposite side is

$$\sqrt{10^2 - 7^2} = \sqrt{51}.$$

Therefore,

$$\tan\theta = \frac{\sqrt{51}}{7}.$$

Hence,

$$\boxed{\tan\left(\cos^{-1}\left(\frac{7}{10}\right)\right) = \frac{\sqrt{51}}{7}.}$$

1.2.12 Comparison with Inverse Sine

Function	Domain	Range
$\sin^{-1} x$	$[-1, 1]$	$[-\frac{\pi}{2}, \frac{\pi}{2}]$
$\cos^{-1} x$	$[-1, 1]$	$[0, \pi]$

Notice that the domains are identical, but the ranges are different.

1.2.13 Common Mistakes

- Confusing

  $$\cos^{-1} x$$

  with

  $$\frac{1}{\cos x}.$$

- Forgetting the principal interval

  $$[0, \pi].$$

- Assuming

  $$\cos^{-1}(\cos\theta) = \theta$$

  for every angle.

- Returning an angle outside

  $$[0, \pi].$$

1.2.14 Practice Problems

Evaluate.

1. $\cos^{-1}(0)$
2. $\cos^{-1}(1)$
3. $\cos^{-1}(-1)$
4. $\cos^{-1}\left(\frac{\sqrt{3}}{2}\right)$
5. $\cos^{-1}\left(-\frac{1}{2}\right)$
6. $\cos^{-1}(\cos(5\pi/3))$
7. $\sin\left(\cos^{-1}\left(\frac{4}{5}\right)\right)$

1.2.15 Answers

1. $\dfrac{\pi}{2}$
2. 0
3. π
4. $\dfrac{\pi}{6}$
5. $\dfrac{2\pi}{3}$
6. $\dfrac{\pi}{3}$
7. $\dfrac{3}{5}$

1.3. The Inverse Tangent Function

1.3.1 Motivation

Unlike the sine and cosine functions, the tangent function has domain

$$(-\infty, \infty),$$

except at odd multiples of $\pi/2$, where it is undefined.
Although tangent repeats every π radians, it is strictly increasing on the interval

$$\left(-\frac{\pi}{2}, \frac{\pi}{2}\right),$$

making it one-to-one on this interval.
Therefore, we restrict the domain of tangent to

$$\left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$$

to define its inverse.
The restricted tangent function is

$$\tan : \left(-\frac{\pi}{2}, \frac{\pi}{2}\right) \to (-\infty, \infty),$$

which is one-to-one and onto.

1.3.2 Definition of the Inverse Tangent Function

Definition 1.10. The inverse tangent function, denoted by

$$\tan^{-1}(x)$$

or

$$\arctan(x),$$

is defined by

$$\boxed{\tan^{-1}(x) = y \iff \tan y = x,}$$

where

$$-\frac{\pi}{2} < y < \frac{\pi}{2}.$$

Thus,

$$\tan^{-1}(x)$$

returns the unique angle in

$$\left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$$

whose tangent equals x.

1.3.3 Domain and Range

Since tangent can take every real value,

$$-\infty < \tan\theta < \infty,$$

the domain of the inverse tangent function is

$$(-\infty, \infty).$$

The range is

$$\boxed{\left(-\frac{\pi}{2}, \frac{\pi}{2}\right).}$$

Therefore,

$$\boxed{\tan^{-1} : \mathbb{R} \to \left(-\frac{\pi}{2}, \frac{\pi}{2}\right).}$$

1.3.4 Important Properties

For every real number x,

$$\boxed{\tan(\tan^{-1} x) = x.}$$

However,

$$\tan^{-1}(\tan\theta) = \theta$$

only if

$$-\frac{\pi}{2} < \theta < \frac{\pi}{2}.$$

Always verify whether the given angle belongs to the principal interval.

1.3.5 Principal Values

The output of

$$\tan^{-1}(x)$$

is called the principal value.

Among all the infinitely many angles whose tangent equals x, inverse tangent returns the one lying in

$$\left(-\frac{\pi}{2}, \frac{\pi}{2}\right).$$

For example,

$$\tan 30^\circ = \tan 210^\circ = \frac{1}{\sqrt{3}}.$$

Nevertheless,

$$\boxed{\tan^{-1}\left(\frac{1}{\sqrt{3}}\right) = 30^\circ,}$$

since 210° is outside the principal interval.

1.3.6 Example 1

Evaluate

$$\tan(\tan^{-1}(1)).$$

Using the identity,

$$\tan(\tan^{-1} x) = x,$$

we obtain

$$\boxed{\tan(\tan^{-1}(1)) = 1.}$$

1.3.7 Example 2

Evaluate

$$\tan^{-1}(\tan 30^\circ).$$

Since

$$30^\circ$$

lies inside

$$(-90^\circ, 90^\circ),$$

the inverse tangent returns the same angle.
Hence,

$$\boxed{\tan^{-1}(\tan 30^\circ) = 30^\circ.}$$

1.3.8 Example 3

Evaluate

$$\tan^{-1}(\tan 210^\circ).$$

First,

$$\tan 210^\circ = \tan 30^\circ = \frac{1}{\sqrt{3}}.$$

Now inverse tangent must return the angle inside

$$(-90^\circ, 90^\circ).$$

The only such angle is

$$30^\circ.$$

Therefore,

$$\boxed{\tan^{-1}(\tan 210^\circ) = 30^\circ.}$$

Notice carefully that the answer is not 210°.

1.3.9 Example 4

Evaluate

$$\tan^{-1}(2).$$

There is no common special angle whose tangent equals 2.
Therefore,

$$\boxed{\tan^{-1}(2)}$$

is already the exact answer.

Using a calculator,

$$\tan^{-1}(2) \approx 63.435^\circ$$

or

$$1.107 \text{ radians.}$$

1.3.10 Example 5

Evaluate

$$\tan^{-1}(-1).$$

We seek the angle in

$$\left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$$

whose tangent equals -1.
That angle is

$$-\frac{\pi}{4}.$$

Therefore,

$$\boxed{\tan^{-1}(-1) = -\frac{\pi}{4}.}$$

1.3.11 Example 6

Evaluate

$$\sin\left(\tan^{-1}\left(\frac{x}{3}\right)\right).$$

This example illustrates one of the most useful techniques in calculus.

Step 1. Introduce an Angle Let

$$\theta = \tan^{-1}\left(\frac{x}{3}\right).$$

Then

$$\tan\theta = \frac{x}{3}.$$

Step 2. Draw a Right Triangle Since

$$\tan\theta = \frac{\text{opposite}}{\text{adjacent}},$$

we choose

$$\text{Opposite} = x,$$

$$\text{Adjacent} = 3.$$

Using the Pythagorean Theorem,

$$\text{Hypotenuse} = \sqrt{x^2+9}.$$

Step 3. Compute the Required Function Now

$$\sin\theta = \frac{\text{opposite}}{\text{hypotenuse}} = \frac{x}{\sqrt{x^2+9}}.$$

Therefore,

$$\boxed{\sin\left(\tan^{-1}\left(\frac{x}{3}\right)\right) = \frac{x}{\sqrt{x^2+9}}.}$$

1.3.12 General Strategy

Whenever an expression contains

$$\sin(\tan^{-1} x), \qquad \cos(\tan^{-1} x), \qquad \tan(\sin^{-1} x),$$

or similar combinations,
follow these steps:

1. Let the inverse trigonometric expression equal an angle θ.
2. Convert the equation into an ordinary trigonometric equation.
3. Draw a right triangle.
4. Use the Pythagorean Theorem to find the missing side.
5. Compute the desired trigonometric function.

1.3.13 Comparison of the Three Inverse Functions

Function	Domain	Range
$\sin^{-1} x$	$[-1, 1]$	$[-\frac{\pi}{2}, \frac{\pi}{2}]$
$\cos^{-1} x$	$[-1, 1]$	$[0, \pi]$
$\tan^{-1} x$	$(-\infty, \infty)$	$(-\frac{\pi}{2}, \frac{\pi}{2})$

1.3.14 Common Mistakes

- Forgetting that inverse tangent returns angles only in
$$\left(-\frac{\pi}{2}, \frac{\pi}{2}\right).$$
- Returning 210° instead of 30° in Example 3.
- Forgetting to draw a right triangle when evaluating expressions such as
$$\sin(\tan^{-1} x).$$
- Mixing degrees and radians in the same calculation.

1.3.15 Practice Problems

Evaluate each expression.

1. $\tan^{-1}(0)$
2. $\tan^{-1}(1)$
3. $\tan^{-1}(\tan 240^\circ)$
4. $\cos\left(\tan^{-1} \frac{4}{5}\right)$
5. $\sin\left(\tan^{-1} \frac{7}{24}\right)$
6. $\sec\left(\tan^{-1} \frac{3}{5}\right)$

1.3.16 Answers

1. 0
2. $\dfrac{\pi}{4}$
3. 60°
4. $\dfrac{5}{\sqrt{41}}$
5. $\dfrac{7}{25}$
6. $\dfrac{\sqrt{34}}{5}$

2. Derivatives and Related Integrals

The derivative formulas for inverse trigonometric functions immediately give standard antiderivatives. For this reason, the derivative and integral formulas are best studied together.

2.1. Derivatives of Inverse Trigonometric Functions

Up to this point we have studied the definitions and properties of the inverse trigonometric functions. We now derive formulas for their derivatives. These derivatives are extremely important because they appear frequently in integration techniques, differential equations, and applications involving related rates.

Throughout this section we shall use the method of **implicit differentiation**. The idea is simple:

1. Introduce a new variable
$$y = f^{-1}(x).$$
2. Rewrite the equation in terms of the original trigonometric function.
3. Differentiate both sides implicitly.
4. Solve for $\dfrac{dy}{dx}$.

2.1.1 Derivative Formulas

The derivatives of the inverse trigonometric functions are summarized below.

Theorem 2.1 (Derivatives of the Inverse Trigonometric Functions)**.** *For all values of x in the domains of the respective functions,*

$$\boxed{\frac{d}{dx}\left(\sin^{-1} x\right) = \frac{1}{\sqrt{1-x^2}}}$$

$$\boxed{\frac{d}{dx}\left(\cos^{-1} x\right) = -\frac{1}{\sqrt{1-x^2}}}$$

$$\boxed{\frac{d}{dx}\left(\tan^{-1} x\right) = \frac{1}{1+x^2}}$$

$$\boxed{\frac{d}{dx}\left(\cot^{-1} x\right) = -\frac{1}{1+x^2}}$$

$$\boxed{\frac{d}{dx}\left(\sec^{-1} x\right) = \frac{1}{|x|\sqrt{x^2-1}}}$$

$$\boxed{\frac{d}{dx}\left(\csc^{-1} x\right) = -\frac{1}{|x|\sqrt{x^2-1}}}$$

Notice that the derivatives of inverse sine and inverse cosine differ only by a negative sign.

2.1.2 Proof of the Derivative of $\sin^{-1}(x)$

We now derive the first formula from scratch.

Suppose

$$y = \sin^{-1}(x).$$

By the definition of inverse sine,

$$\sin y = x.$$

Since y is a function of x, differentiate both sides with respect to x.
Using the Chain Rule,

$$\frac{d}{dx}(\sin y) = \cos y \frac{dy}{dx}.$$

Therefore,

$$\cos y \frac{dy}{dx} = 1.$$

Solving for the derivative gives

$$\frac{dy}{dx} = \frac{1}{\cos y}.$$

At this point the derivative is still written in terms of the variable y. Our goal is to express everything in terms of x.

2.1.3 Using a Right Triangle

Since

$$\sin y = x,$$

we interpret sine as

$$\sin y = \frac{\text{opposite}}{\text{hypotenuse}}.$$

Choose

$$\text{Opposite} = x, \qquad \text{Hypotenuse} = 1.$$

By the Pythagorean Theorem,

$$\text{Adjacent} = \sqrt{1 - x^2}.$$

Therefore,

$$\cos y = \frac{\sqrt{1 - x^2}}{1} = \sqrt{1 - x^2}.$$

Substituting into the derivative,

$$\frac{dy}{dx} = \frac{1}{\sqrt{1 - x^2}}.$$

Hence,

$$\boxed{\frac{d}{dx}(\sin^{-1} x) = \frac{1}{\sqrt{1 - x^2}}.}$$

2.1.4 Why is the Square Root Positive?

You might wonder why we chose

$$\sqrt{1-x^2}$$

instead of

$$-\sqrt{1-x^2}.$$

Recall that

$$y = \sin^{-1}(x)$$

always satisfies

$$-\frac{\pi}{2} \leq y \leq \frac{\pi}{2}.$$

On this interval,

$$\cos y \geq 0.$$

Therefore,

$$\cos y = \sqrt{1-x^2},$$

never the negative square root.
This is one reason why restricting the range of inverse sine is so important.

2.1.5 Example 1

Differentiate

$$f(x) = \sin^{-1}(3x).$$

Use the Chain Rule.
Let

$$u = 3x.$$

Then

$$f(x) = \sin^{-1}(u).$$

Differentiate.

$$f'(x) = \frac{1}{\sqrt{1-u^2}} \cdot \frac{du}{dx}.$$

Since

$$\frac{du}{dx} = 3,$$

we obtain

$$\boxed{f'(x) = \frac{3}{\sqrt{1-9x^2}}.}$$

2.1.6 Example 2

Differentiate

$$f(x) = 5\sin^{-1}(x).$$

The constant multiple rule gives

$$f'(x) = 5 \cdot \frac{1}{\sqrt{1-x^2}}.$$

Therefore,

$$\boxed{f'(x) = \frac{5}{\sqrt{1-x^2}}.}$$

2.1.7 Derivative of $\cos^{-1}(x)$

Rather than proving this separately, observe the identity

$$\sin^{-1}(x) + \cos^{-1}(x) = \frac{\pi}{2}.$$

Differentiate both sides.

$$\frac{d}{dx}(\sin^{-1} x) + \frac{d}{dx}(\cos^{-1} x) = 0.$$

Substitute the derivative of inverse sine.

$$\frac{1}{\sqrt{1-x^2}} + \frac{d}{dx}(\cos^{-1} x) = 0.$$

Hence,

$$\boxed{\frac{d}{dx}(\cos^{-1} x) = -\frac{1}{\sqrt{1-x^2}}.}$$

2.1.8 Proof of the Derivative of $\tan^{-1}(x)$

Let

$$y = \tan^{-1}(x).$$

Then

$$\tan y = x.$$

Differentiate both sides.
Using the Chain Rule,

$$\sec^2 y \frac{dy}{dx} = 1.$$

Therefore,

$$\frac{dy}{dx} = \frac{1}{\sec^2 y}.$$

Recall the identity

$$\sec^2 y = 1 + \tan^2 y.$$

Since

$$\tan y = x,$$

we have

$$\sec^2 y = 1 + x^2.$$

Hence,

$$\boxed{\frac{d}{dx}(\tan^{-1} x) = \frac{1}{1+x^2}.}$$

2.1.9 Example 3

Differentiate

$$y = \tan^{-1}(4x^2).$$

Let

$$u = 4x^2.$$

Then

$$y = \tan^{-1}(u).$$

Differentiate.

$$\frac{dy}{dx} = \frac{1}{1+u^2} \cdot \frac{du}{dx}.$$

Since

$$\frac{du}{dx} = 8x,$$

we obtain

$$\boxed{\frac{dy}{dx} = \frac{8x}{1+16x^4}.}$$

2.1.10 Summary

The three most important derivative formulas to remember are

$$\boxed{\frac{d}{dx}(\sin^{-1} x) = \frac{1}{\sqrt{1-x^2}}}$$

$$\boxed{\frac{d}{dx}(\cos^{-1} x) = -\frac{1}{\sqrt{1-x^2}}}$$

$$\boxed{\frac{d}{dx}(\tan^{-1} x) = \frac{1}{1+x^2}}$$

Every other inverse trigonometric derivative can be derived using similar implicit differentiation techniques.

2.2. Related Integrals

In the previous section, we derived the derivatives of the inverse trigonometric functions. Recall one of the fundamental ideas of calculus:

> Differentiation and integration are inverse operations.

Therefore, every derivative formula immediately gives rise to an integration formula.

For example, since

$$\frac{d}{dx}\left(\sin^{-1} x\right) = \frac{1}{\sqrt{1-x^2}},$$

it follows immediately that

$$\boxed{\int \frac{dx}{\sqrt{1-x^2}} = \sin^{-1} x + C.}$$

Similarly,

$$\frac{d}{dx}\left(\tan^{-1} x\right) = \frac{1}{1+x^2}$$

implies

$$\boxed{\int \frac{dx}{1+x^2} = \tan^{-1} x + C.}$$

2.2.1 Standard Inverse Trigonometric Integrals

The following formulas should be memorized.

$$\boxed{\int \frac{dx}{\sqrt{1-x^2}} = \sin^{-1} x + C}$$

$$\boxed{\int \frac{-dx}{\sqrt{1-x^2}} = \cos^{-1} x + C}$$

$$\boxed{\int \frac{dx}{1+x^2} = \tan^{-1} x + C}$$

These three formulas appear frequently throughout Calculus II.

2.2.2 Generalized Forms

The previous formulas can be generalized by introducing a positive constant a.

Inverse Sine Form Suppose

$$u = \frac{x}{a}.$$

Then

$$dx = a\,du.$$

Hence,

$$\begin{aligned}\int \frac{dx}{\sqrt{a^2 - x^2}} &= \int \frac{a\,du}{\sqrt{a^2 - a^2u^2}} \\ &= \int \frac{du}{\sqrt{1-u^2}} \\ &= \sin^{-1}(u) + C.\end{aligned}$$

Substituting back,

$$\boxed{\int \frac{dx}{\sqrt{a^2 - x^2}} = \sin^{-1}\left(\frac{x}{a}\right) + C.}$$

Inverse Tangent Form Similarly,

$$u = \frac{x}{a}$$

gives

$$dx = a\,du.$$

Then

$$\begin{aligned}\int \frac{dx}{a^2 + x^2} &= \int \frac{a\,du}{a^2(1+u^2)} \\ &= \frac{1}{a}\int \frac{du}{1+u^2} \\ &= \frac{1}{a}\tan^{-1}(u) + C.\end{aligned}$$

Therefore,

$$\boxed{\int \frac{dx}{a^2 + x^2} = \frac{1}{a}\tan^{-1}\left(\frac{x}{a}\right) + C.}$$

2.2.3 Recognizing Inverse Trigonometric Integrals

Whenever you encounter an integral, first ask whether it matches one of the standard inverse trigonometric forms.

Look for

$$\frac{1}{\sqrt{a^2 - x^2}},$$

or

$$\frac{1}{a^2 + x^2}.$$

If necessary,

- factor constants,
- complete the square,
- perform a simple substitution,

before deciding whether the integral has an inverse trigonometric antiderivative.

2.2.4 Example 1

Evaluate

$$\int \frac{dx}{\sqrt{1 - x^2}}.$$

Since this matches the standard inverse sine form exactly,

$$\boxed{\int \frac{dx}{\sqrt{1 - x^2}} = \sin^{-1}(x) + C.}$$

2.2.5 Example 2

Evaluate

$$\int \frac{dx}{1 + x^2}.$$

Again,
this is already one of our standard formulas.
Hence,

$$\boxed{\int \frac{dx}{1 + x^2} = \tan^{-1}(x) + C.}$$

2.2.6 Example 3

Evaluate

$$\int \frac{dx}{9 + x^2}.$$

Notice that

$$9 + x^2 = 3^2 + x^2.$$

Using the generalized inverse tangent formula,

$$\boxed{\int \frac{dx}{9 + x^2} = \frac{1}{3} \tan^{-1}\left(\frac{x}{3}\right) + C.}$$

2.2.7 Example 4

Evaluate

$$\int \frac{5}{25 + x^2}\, dx.$$

Factor the constant carefully.
Using

$$a = 5,$$

we obtain

$$\boxed{\int \frac{5}{25 + x^2} dx = \tan^{-1}\left(\frac{x}{5}\right) + C.}$$

2.2.8 Example 5

Evaluate

$$\int \frac{-dx}{\sqrt{1 - x^2}}.$$

Since

$$\frac{d}{dx}(\cos^{-1} x) = -\frac{1}{\sqrt{1 - x^2}},$$

we immediately have

$$\boxed{\int \frac{-dx}{\sqrt{1 - x^2}} = \cos^{-1}(x) + C.}$$

Alternatively,
because

$$\cos^{-1}(x) = \frac{\pi}{2} - \sin^{-1}(x),$$

the answer may also be written as

$$-\sin^{-1}(x) + C.$$

2.2.9 Example 6

Evaluate

$$\int \frac{2}{4 + x^2} dx.$$

Notice

$$4 = 2^2.$$

Thus,

$$\int \frac{2}{4+x^2}dx = 2\left(\frac{1}{2}\tan^{-1}\left(\frac{x}{2}\right)\right) + C$$
$$= \boxed{\tan^{-1}\left(\frac{x}{2}\right) + C.}$$

2.2.10 Common Mistakes

Students often make the following errors.

1. Forgetting the factor
$$\frac{1}{a}$$
in
$$\int \frac{dx}{a^2+x^2}.$$
2. Confusing
$$\sqrt{a^2-x^2}$$
with
$$a^2-x^2.$$
3. Using an inverse trigonometric formula when a simple u-substitution would be easier.
4. Forgetting to simplify constants before integrating.

2.2.11 Practice Problems

Evaluate.

1. $$\int \frac{dx}{16+x^2}$$
2. $$\int \frac{dx}{\sqrt{9-x^2}}$$
3. $$\int \frac{3}{9+x^2}dx$$
4. $$\int \frac{-dx}{\sqrt{4-x^2}}$$
5. $$\int \frac{dx}{1+9x^2}$$

2.2.12 Answers

1. $$\frac{1}{4}\tan^{-1}\left(\frac{x}{4}\right)+C$$

2. $$\sin^{-1}\left(\frac{x}{3}\right)+C$$

3. $$\tan^{-1}\left(\frac{x}{3}\right)+C$$

4. $$-\sin^{-1}\left(\frac{x}{2}\right)+C$$

   or

   $$\cos^{-1}\left(\frac{x}{2}\right)+C$$

5. Let

   $$u=3x.$$

   Then

   $$\boxed{\frac{1}{3}\tan^{-1}(3x)+C.}$$

3. Trigonometric Substitution: Motivation and Setup

3.1. Introduction

In previous chapters we learned several methods of integration, including

- substitution,
- integration by parts,
- partial fractions,
- trigonometric identities.

Unfortunately, these techniques are not sufficient for every integral.
For example,

$$\int \sqrt{9-x^2}\,dx,$$

or

$$\int \frac{dx}{\sqrt{x^2+4}},$$

cannot be evaluated easily using ordinary substitution.
The difficulty comes from the square root.

The purpose of **trigonometric substitution** is to transform these radicals into perfect squares by exploiting the fundamental identities of trigonometry.

3.2. The Fundamental Trigonometric Identities

The three identities used throughout this chapter are

$$\boxed{\sin^2\theta + \cos^2\theta = 1}$$

$$\boxed{1 + \tan^2\theta = \sec^2\theta}$$

$$\boxed{\sec^2\theta - 1 = \tan^2\theta}$$

Notice that each identity contains a perfect square.
This is precisely why trigonometric substitution works.

3.3. Motivation

Suppose we wish to simplify

$$\sqrt{a^2 - x^2}.$$

If we let

$$x = a\sin\theta,$$

then

$$x^2 = a^2\sin^2\theta.$$

Therefore,

$$\begin{aligned}\sqrt{a^2 - x^2} &= \sqrt{a^2 - a^2\sin^2\theta}\\ &= \sqrt{a^2(1-\sin^2\theta)}\\ &= a\sqrt{\cos^2\theta}\\ &= a\cos\theta.\end{aligned}$$

The complicated radical has disappeared.
Instead of integrating a square root, we now integrate a trigonometric function.

3.4. The Three Standard Substitutions

There are only three substitutions that need to be memorized.

Radical	**Substitution**	**Identity**
$\sqrt{a^2 - x^2}$	$x = a\sin\theta$	$1 - \sin^2\theta = \cos^2\theta$
$\sqrt{a^2 + x^2}$	$x = a\tan\theta$	$1 + \tan^2\theta = \sec^2\theta$
$\sqrt{x^2 - a^2}$	$x = a\sec\theta$	$\sec^2\theta - 1 = \tan^2\theta$

These substitutions should be memorized together with the identities that justify them.

3.5. Why These Particular Choices?

Students often ask,

> Why do we choose sine in one problem and tangent in another?

The answer is simple.

Our goal is to eliminate the square root.

Each substitution is chosen so that the expression inside the radical becomes one of the fundamental trigonometric identities.

For example,

$$a^2 - x^2$$

becomes

$$a^2 - a^2 \sin^2\theta = a^2(1 - \sin^2\theta),$$

which simplifies immediately.

Likewise,

$$a^2 + x^2$$

becomes

$$a^2 + a^2 \tan^2\theta = a^2(1 + \tan^2\theta) = a^2 \sec^2\theta.$$

Finally,

$$x^2 - a^2$$

becomes

$$a^2 \sec^2\theta - a^2 = a^2(\sec^2\theta - 1) = a^2 \tan^2\theta.$$

3.6. Choosing the Correct Substitution

The following decision tree is extremely useful.

$$\boxed{\begin{array}{c}\sqrt{a^2 - x^2}\\ \Downarrow \\ x = a\sin\theta\end{array}}$$

$$\boxed{\begin{array}{c}\sqrt{a^2 + x^2}\\ \Downarrow \\ x = a\tan\theta\end{array}}$$

$$\boxed{\begin{array}{c}\sqrt{x^2 - a^2}\\ \Downarrow \\ x = a\sec\theta\end{array}}$$

3.7. General Procedure

Almost every trigonometric substitution problem follows the same six steps.

1. Identify the radical.
2. Choose the correct substitution.
3. Differentiate to obtain dx.
4. Substitute every occurrence of x and dx.
5. Simplify using trigonometric identities.
6. Integrate.
7. Draw a right triangle.
8. Convert the answer back to the variable x.

Many students forget the final step.
Never leave the answer in terms of θ.

3.8. Example 0: Simplifying a Radical

Suppose

$$x = 5\sin\theta.$$

Then

$$dx = 5\cos\theta\, d\theta.$$

Now simplify

$$\sqrt{25 - x^2}.$$

Substitute.

$$\begin{aligned}\sqrt{25 - x^2} &= \sqrt{25 - 25\sin^2\theta} \\ &= \sqrt{25(1 - \sin^2\theta)} \\ &= 5\sqrt{\cos^2\theta} \\ &= 5\cos\theta.\end{aligned}$$

Observe that the square root disappears completely.
This illustrates the central idea of trigonometric substitution.

3.9. Common Mistakes

Students frequently make the following errors.

1. Choosing the wrong substitution.
2. Forgetting to differentiate to obtain dx.
3. Forgetting to replace every occurrence of x.
4. Leaving the answer in terms of θ.
5. Forgetting to draw a right triangle during back-substitution.

3.10. Summary

The purpose of trigonometric substitution is not to make an integral look more complicated.

Instead, it converts difficult radicals into simple trigonometric expressions using one of the three fundamental identities.

Once the substitution is made, the resulting integral usually becomes much easier to evaluate.

4. Worked Examples and Technique Selection

The examples below illustrate both when trigonometric substitution is useful and when a simpler method should be chosen instead.

4.1. Evaluating $\int \sqrt{1-x^2}\,dx$

We now illustrate the method of trigonometric substitution by evaluating one of the most important integrals in Calculus II.

$$I = \int \sqrt{1-x^2}\,dx.$$

Notice that the radical has the form

$$\sqrt{a^2 - x^2},$$

where

$$a = 1.$$

From our table of substitutions, we choose

$$\boxed{x = \sin\theta.}$$

4.1.1 Step 1: Differentiate the Substitution

Differentiate both sides with respect to θ.

$$x = \sin\theta$$

implies

$$dx = \cos\theta\,d\theta.$$

This differential must also be substituted into the integral.

4.1.2 Step 2: Simplify the Radical

Now simplify

$$\sqrt{1-x^2}.$$

Since

$$x = \sin\theta,$$

we obtain

$$\sqrt{1-x^2} = \sqrt{1-\sin^2\theta}.$$

Using the Pythagorean identity,

$$\sin^2\theta + \cos^2\theta = 1,$$

we have

$$1 - \sin^2\theta = \cos^2\theta.$$

Therefore,

$$\sqrt{1-\sin^2\theta} = \sqrt{\cos^2\theta}.$$

Because

$$-\frac{\pi}{2} \le \theta \le \frac{\pi}{2},$$

we know that

$$\cos\theta \ge 0.$$

Hence,

$$\boxed{\sqrt{\cos^2\theta} = \cos\theta.}$$

Thus,

$$\boxed{\sqrt{1-x^2} = \cos\theta.}$$

4.1.3 Step 3: Rewrite the Integral

Substitute both expressions into the integral.

$$\begin{aligned} I &= \int \sqrt{1-\sin^2\theta}\, \cos\theta\, d\theta \\ &= \int \cos\theta \cos\theta\, d\theta \\ &= \int \cos^2\theta\, d\theta. \end{aligned}$$

The original radical has disappeared completely.

4.1.4 Step 4: Integrate $\cos^2\theta$

Unfortunately,

$$\int \cos^2\theta\, d\theta$$

cannot be integrated directly.
Instead, we use the half-angle identity

$$\boxed{\cos^2\theta = \frac{1+\cos 2\theta}{2}.}$$

Therefore,

$$\begin{aligned} I &= \int \frac{1+\cos 2\theta}{2}\, d\theta \\ &= \frac{1}{2}\int (1+\cos 2\theta)\, d\theta. \end{aligned}$$

Distribute the integral.

$$I = \frac{1}{2}\int 1\, d\theta + \frac{1}{2}\int \cos 2\theta\, d\theta.$$

Integrate each term separately.
The first integral is

$$\int 1\, d\theta = \theta.$$

For the second integral,
let

$$u = 2\theta, \qquad du = 2\, d\theta.$$

Hence,

$$\int \cos 2\theta\, d\theta = \frac{1}{2}\sin 2\theta.$$

Substituting,

$$\boxed{I = \frac{1}{2}\theta + \frac{1}{4}\sin 2\theta + C.}$$

4.1.5 Step 5: Simplify the Double Angle

Recall the identity

$$\boxed{\sin 2\theta = 2\sin\theta\cos\theta.}$$

Substitute into the previous answer.

$$\begin{aligned} I &= \frac{1}{2}\theta + \frac{1}{4}(2\sin\theta\cos\theta) + C \\ &= \frac{1}{2}\theta + \frac{1}{2}\sin\theta\cos\theta + C. \end{aligned}$$

4.1.6 Step 6: Draw a Right Triangle

Our substitution was

$$x = \sin\theta.$$

Therefore,

$$\sin\theta = \frac{x}{1}.$$

Interpret sine as

$$\frac{\text{opposite}}{\text{hypotenuse}}.$$

Construct the following right triangle.

$$\text{Hypotenuse} = 1,$$

$$\text{Opposite} = x,$$

$$\text{Adjacent} = \sqrt{1-x^2}.$$

From this triangle,

$$\boxed{\cos\theta = \sqrt{1-x^2}.}$$

Also,

$$\boxed{\theta = \sin^{-1}(x).}$$

4.1.7 Step 7: Back-Substitute

Replace every trigonometric quantity with an expression involving x.

Since

$$\theta = \sin^{-1}(x),$$

and

$$\sin\theta = x,$$

$$\cos\theta = \sqrt{1-x^2},$$

we obtain

$$I = \frac{1}{2}\sin^{-1}(x) + \frac{1}{2}x\sqrt{1-x^2} + C.$$

Therefore,

$$\boxed{\int \sqrt{1-x^2}\,dx = \frac{1}{2}\sin^{-1}(x) + \frac{1}{2}x\sqrt{1-x^2} + C.}$$

4.1.8 Verification

Differentiate the answer.

$$\frac{d}{dx}\left(\frac{1}{2}\sin^{-1}(x) + \frac{1}{2}x\sqrt{1-x^2}\right).$$

Using the Product Rule and the derivative of $\sin^{-1}(x)$,
one finds that all terms simplify to

$$\boxed{\sqrt{1-x^2},}$$

confirming that the antiderivative is correct.

4.1.9 Geometric Interpretation

The substitution

$$x = \sin\theta$$

transforms a semicircle

$$x^2 + y^2 = 1$$

into a right triangle whose sides are

$$x, \qquad \sqrt{1-x^2}, \qquad 1.$$

This geometric interpretation explains why trigonometric substitution is so effective when radicals of the form

$$\sqrt{a^2 - x^2}$$

appear.

4.1.10 Important Observations

1. Always identify the radical before choosing the substitution.
2. Never forget to substitute for dx.
3. Use trigonometric identities to eliminate radicals.
4. Convert every trigonometric quantity back to x before giving the final answer.
5. Never leave the answer in terms of θ.

4.2. Choosing the Appropriate Integration Technique

One of the most important skills in Calculus II is selecting the correct integration method. Trigonometric substitution is a powerful technique, but it should not be used unless it is truly necessary.

Before applying trigonometric substitution, ask yourself the following questions.

1. Can a simple u-substitution be used?
2. Can the integrand be simplified algebraically?
3. Is integration by parts appropriate?
4. Is partial fraction decomposition possible?
5. Does the integrand contain a radical of the form

$$\sqrt{a^2 - x^2}, \qquad \sqrt{a^2 + x^2}, \qquad \sqrt{x^2 - a^2}?$$

Only after eliminating the simpler methods should trigonometric substitution be considered.

4.3. A Simpler u-Substitution Example

At first glance the radical

$$\sqrt{x^2+4}$$

suggests using the substitution

$$x = 2\tan\theta.$$

Although this method works, it is unnecessarily complicated.
Instead, notice that the numerator is the derivative of the expression inside the square root.

4.3.1 Step 1: Choose a u-Substitution

Let

$$u = x^2 + 4.$$

Differentiate.

$$du = 2x\,dx.$$

Therefore,

$$x\,dx = \frac{1}{2}\,du.$$

4.3.2 Step 2: Rewrite the Integral

Substitute into the integral.

$$\begin{aligned} I &= \int \frac{x}{\sqrt{x^2+4}} dx \\ &= \frac{1}{2}\int u^{-1/2}\,du. \end{aligned}$$

4.3.3 Step 3: Integrate

Using the Power Rule,

$$\int u^{-1/2} du = 2u^{1/2}.$$

Hence,

$$\begin{aligned} I &= \frac{1}{2}(2u^{1/2}) + C \\ &= u^{1/2} + C. \end{aligned}$$

Substitute back.

$$\boxed{\int \frac{x}{\sqrt{x^2+4}} dx = \sqrt{x^2+4} + C.}$$

4.3.4 Verification

Differentiate

$$\sqrt{x^2+4}.$$

Using the Chain Rule,

$$\frac{d}{dx}\left(\sqrt{x^2+4}\right) = \frac{1}{2\sqrt{x^2+4}}(2x) = \frac{x}{\sqrt{x^2+4}}.$$

Therefore,

$$\boxed{\frac{d}{dx}\left(\sqrt{x^2+4}\right) = \frac{x}{\sqrt{x^2+4}},}$$

which confirms our answer.

4.4. Why Trigonometric Substitution Is Not the Best Choice

Suppose we instead choose

$$x = 2\tan\theta.$$

Then

$$dx = 2\sec^2\theta\, d\theta,$$

and

$$\sqrt{x^2+4} = 2\sec\theta.$$

The integral becomes

$$\int \frac{2\tan\theta}{2\sec\theta}(2\sec^2\theta)d\theta = 2\int \tan\theta\sec\theta d\theta.$$

Since

$$\frac{d}{d\theta}(\sec\theta) = \sec\theta\tan\theta,$$

the integral becomes

$$2\sec\theta + C.$$

Returning to x,

$$\sec\theta = \frac{\sqrt{x^2+4}}{2},$$

which gives

$$2\left(\frac{\sqrt{x^2+4}}{2}\right) = \sqrt{x^2+4}.$$

Thus both methods produce the same answer,

$$\boxed{\sqrt{x^2+4} + C,}$$

but the ordinary substitution requires only three lines of work, whereas the trigonometric substitution requires several additional steps.

4.4.1 Lesson

This example illustrates an important principle.

> The correct integration technique is not always the most sophisticated one. Choose the simplest method that works.

A good calculus student recognizes when trigonometric substitution is appropriate and, equally importantly, when it is unnecessary.

4.5. Integrals Involving $\sqrt{a^2 + x^2}$

In this section we study integrals containing radicals of the form

$$\sqrt{a^2 + x^2}.$$

Whenever this expression appears, we use the substitution

$$\boxed{x = a \tan\theta.}$$

The reason is the Pythagorean identity

$$1 + \tan^2\theta = \sec^2\theta.$$

This identity converts the square root into a single trigonometric function.

4.5.1 Example

Evaluate

$$I = \int \frac{x^2}{\sqrt{x^2 + 4}}\, dx.$$

Notice that

$$x^2 + 4 = x^2 + 2^2,$$

so the correct substitution is

$$x = 2\tan\theta.$$

4.5.2 Step 1: Differentiate

Differentiate both sides.

$$dx = 2\sec^2\theta\, d\theta.$$

4.5.3 Step 2: Simplify the Radical

Substitute

$$x = 2\tan\theta.$$

Then

$$\begin{aligned}\sqrt{x^2+4} &= \sqrt{4\tan^2\theta+4}\\ &= \sqrt{4(\tan^2\theta+1)}\\ &= \sqrt{4\sec^2\theta}\\ &= 2\sec\theta.\end{aligned}$$

Notice that the square root disappears completely.

4.5.4 Step 3: Rewrite the Integral

Substitute every occurrence of x and dx.

$$\begin{aligned}I &= \int \frac{4\tan^2\theta}{2\sec\theta}(2\sec^2\theta)\,d\theta\\ &= 4\int \tan^2\theta\sec\theta\,d\theta.\end{aligned}$$

4.5.5 Step 4: Rewrite the Integrand

Recall that

$$\tan^2\theta = \sec^2\theta - 1.$$

Therefore,

$$\begin{aligned}I &= 4\int (\sec^2\theta-1)\sec\theta\,d\theta\\ &= 4\int (\sec^3\theta - \sec\theta)\,d\theta.\end{aligned}$$

The integral has now been reduced to two standard secant integrals.

4.5.6 Step 5: Evaluate the Secant Integrals

Recall the standard formulas

$$\boxed{\int \sec\theta\,d\theta = \ln|\sec\theta+\tan\theta| + C,}$$

and

$$\boxed{\int \sec^3\theta\,d\theta = \frac{1}{2}\sec\theta\tan\theta + \frac{1}{2}\ln|\sec\theta+\tan\theta| + C.}$$

Using these formulas,

$$\begin{aligned}I &= 4\left[\frac{1}{2}\sec\theta\tan\theta + \frac{1}{2}\ln|\sec\theta+\tan\theta| - \ln|\sec\theta+\tan\theta|\right] + C\\ &= 2\sec\theta\tan\theta - 2\ln|\sec\theta+\tan\theta| + C.\end{aligned}$$

4.5.7 Step 6: Draw the Reference Triangle

Since

$$\tan\theta = \frac{x}{2},$$

construct a right triangle with

$$\text{Opposite} = x,$$

$$\text{Adjacent} = 2.$$

The hypotenuse is

$$\sqrt{x^2+4}.$$

Therefore,

$$\sec\theta = \frac{\sqrt{x^2+4}}{2},$$

and

$$\tan\theta = \frac{x}{2}.$$

4.5.8 Step 7: Back-Substitute

First,

$$2\sec\theta\tan\theta = 2\left(\frac{\sqrt{x^2+4}}{2}\right)\left(\frac{x}{2}\right) = \frac{x\sqrt{x^2+4}}{2}.$$

Next,

$$\sec\theta + \tan\theta = \frac{\sqrt{x^2+4}}{2} + \frac{x}{2} = \frac{x+\sqrt{x^2+4}}{2}.$$

Since

$$\ln\left(\frac{u}{2}\right) = \ln u - \ln 2,$$

the constant

$$\ln 2$$

is absorbed into the constant of integration.
Hence

$$\boxed{I = \frac{x\sqrt{x^2+4}}{2} - 2\ln\left|x+\sqrt{x^2+4}\right| + C.}$$

4.5.9 Verification

Differentiate the answer.

After applying the Product Rule and Chain Rule, all logarithmic terms cancel, leaving

$$\boxed{\frac{x^2}{\sqrt{x^2+4}}.}$$

Thus the antiderivative is correct.

4.5.10 Remark

Notice how the substitution

$$x = 2\tan\theta$$

completely eliminated the square root by using the identity

$$1+\tan^2\theta = \sec^2\theta.$$

This is the defining feature of every trigonometric substitution problem.

4.5.11 Common Errors

Students frequently make the following mistakes.

1. Forgetting to replace dx.
2. Writing

$$\sqrt{4\sec^2\theta} = 4\sec\theta$$

instead of

$$2\sec\theta.$$

3. Forgetting to use

$$\tan^2\theta = \sec^2\theta - 1.$$

4. Leaving the answer in terms of θ.
5. Forgetting that logarithms differing by a constant are absorbed into the integration constant.

4.6. Integrals Involving $\sqrt{x^2-a^2}$

The final standard trigonometric substitution is used whenever the integrand contains the radical

$$\sqrt{x^2-a^2}.$$

The appropriate substitution is

$$\boxed{x = a\sec\theta.}$$

This substitution is motivated by the identity

$$\boxed{\sec^2\theta - 1 = \tan^2\theta.}$$

4.6.1 Example

Evaluate

$$I = \int \frac{dx}{x^2\sqrt{x^2-4}}.$$

Since

$$x^2 - 4 = x^2 - 2^2,$$

we choose

$$\boxed{x = 2\sec\theta.}$$

4.6.2 Step 1: Differentiate

Differentiate both sides.

$$dx = 2\sec\theta\tan\theta\, d\theta.$$

4.6.3 Step 2: Simplify the Radical

Substitute

$$x = 2\sec\theta.$$

Then

$$\begin{aligned}
\sqrt{x^2-4} &= \sqrt{4\sec^2\theta - 4} \\
&= \sqrt{4(\sec^2\theta - 1)} \\
&= \sqrt{4\tan^2\theta} \\
&= 2\tan\theta.
\end{aligned}$$

Since

$$0 \le \theta < \frac{\pi}{2},$$

we have

$$\tan\theta \ge 0,$$

so

$$\sqrt{\tan^2\theta} = \tan\theta.$$

4.6.4 Step 3: Rewrite the Integral

Replace every occurrence of x.

Since

$$x^2 = 4\sec^2\theta,$$

we obtain

$$\begin{aligned} I &= \int \frac{2\sec\theta\tan\theta}{4\sec^2\theta(2\tan\theta)}\,d\theta \\ &= \frac{1}{4}\int \frac{1}{\sec\theta}\,d\theta. \end{aligned}$$

Because

$$\frac{1}{\sec\theta} = \cos\theta,$$

the integral becomes

$$\boxed{I = \frac{1}{4}\int \cos\theta\,d\theta.}$$

4.6.5 Step 4: Integrate

Since

$$\int \cos\theta\,d\theta = \sin\theta,$$

we obtain

$$I = \frac{1}{4}\sin\theta + C.$$

4.6.6 Step 5: Draw the Reference Triangle

Our substitution was

$$x = 2\sec\theta.$$

Therefore,

$$\sec\theta = \frac{x}{2}.$$

Recall that

$$\sec\theta = \frac{\text{Hypotenuse}}{\text{Adjacent}}.$$

Construct a right triangle.

$$\text{Adjacent} = 2,$$

$$\text{Hypotenuse} = x.$$

The remaining side is

$$\sqrt{x^2-4}.$$

Therefore,

$$\sin\theta = \frac{\sqrt{x^2-4}}{x}.$$

4.6.7 Step 6: Back-Substitute

Replace

$$\sin\theta = \frac{\sqrt{x^2-4}}{x}.$$

Hence,

$$\boxed{I = \frac{\sqrt{x^2-4}}{4x} + C.}$$

4.6.8 Verification

Differentiate

$$\frac{\sqrt{x^2-4}}{4x}.$$

Using the Quotient Rule,

$$\begin{aligned}
\frac{d}{dx}\left(\frac{\sqrt{x^2-4}}{4x}\right) &= \frac{1}{4}\cdot\frac{x\left(\frac{x}{\sqrt{x^2-4}}\right)-\sqrt{x^2-4}}{x^2} \\
&= \frac{1}{4}\cdot\frac{x^2-(x^2-4)}{x^2\sqrt{x^2-4}} \\
&= \frac{1}{4}\cdot\frac{4}{x^2\sqrt{x^2-4}} \\
&= \boxed{\frac{1}{x^2\sqrt{x^2-4}}.}
\end{aligned}$$

Therefore the antiderivative is correct.

4.6.9 Why We Use Secant

Suppose instead we attempted

$$x = 2\sin\theta.$$

Then

$$x^2-4 = 4\sin^2\theta - 4,$$

which becomes

$$4(\sin^2\theta - 1),$$

and no standard identity simplifies this expression.
Similarly,

$$x = 2\tan\theta$$

produces

$$4(\tan^2\theta - 1),$$

which also fails to simplify.
Only

$$x = 2\sec\theta$$

produces

$$4(\sec^2\theta - 1) = 4\tan^2\theta,$$

which eliminates the radical immediately.

4.6.10 Comparison of the Three Trigonometric Substitutions

Radical	Substitution	Identity
$\sqrt{a^2 - x^2}$	$x = a\sin\theta$	$1 - \sin^2\theta = \cos^2\theta$
$\sqrt{a^2 + x^2}$	$x = a\tan\theta$	$1 + \tan^2\theta = \sec^2\theta$
$\sqrt{x^2 - a^2}$	$x = a\sec\theta$	$\sec^2\theta - 1 = \tan^2\theta$

This table summarizes all possible trigonometric substitutions encountered in Calculus II.

4.6.11 Common Mistakes

1. Forgetting to square the substitution.
2. Incorrectly simplifying
$$\sqrt{4\tan^2\theta}.$$
3. Forgetting that
$$\frac{1}{\sec\theta} = \cos\theta.$$
4. Forgetting to construct the reference triangle before back-substitution.
5. Leaving the final answer in terms of θ.

5. Strategy, Common Errors, and Formula Summary

5.1. Strategy for Solving Trigonometric Substitution Problems

Many students memorize the substitution table without understanding when to apply each substitution. The following strategy can be used for every trigonometric substitution problem.

5.1.1 Step 1. Examine the Radical

The first step is to identify the expression inside the square root.

Three cases are possible.

$$\sqrt{a^2 - x^2}$$

$$\sqrt{a^2 + x^2}$$

$$\sqrt{x^2 - a^2}$$

Once the radical has been identified, the correct substitution follows immediately.

5.1.2 Step 2. Choose the Appropriate Substitution

Radical	**Substitution**
$\sqrt{a^2 - x^2}$	$x = a\sin\theta$
$\sqrt{a^2 + x^2}$	$x = a\tan\theta$
$\sqrt{x^2 - a^2}$	$x = a\sec\theta$

Never mix these substitutions.

5.1.3 Step 3. Differentiate

Differentiate your substitution.

For example,

$$x = a\sin\theta$$

gives

$$dx = a\cos\theta\, d\theta.$$

Likewise,

$$x = a\tan\theta$$

gives

$$dx = a\sec^2\theta\, d\theta,$$

and

$$x = a\sec\theta$$

gives

$$dx = a\sec\theta\tan\theta\, d\theta.$$

5.1.4 Step 4. Rewrite the Entire Integral

Replace

- every occurrence of x,
- every occurrence of dx,
- every radical.

Do not replace only part of the integral.

5.1.5 Step 5. Simplify

Use the Pythagorean identities

$$1 - \sin^2\theta = \cos^2\theta,$$

$$1 + \tan^2\theta = \sec^2\theta,$$

$$\sec^2\theta - 1 = \tan^2\theta.$$

The purpose of the substitution is to eliminate the radical.

5.1.6 Step 6. Integrate

After simplification the remaining integral usually involves

$$\sin\theta, \qquad \cos\theta, \qquad \tan\theta, \qquad \sec\theta.$$

Use the integration techniques developed in previous chapters.

5.1.7 Step 7. Back-Substitute

This step is frequently forgotten.
Construct a right triangle using the substitution.
Convert

$$\sin\theta, \quad \cos\theta, \quad \tan\theta, \quad \sec\theta$$

back into expressions involving x.
The final answer should never contain the variable θ.

5.2. Decision Tree

Before using trigonometric substitution, ask the following questions.

Does the integrand contain a radical?

$\Downarrow$

No $\rightarrow$ Use another method.

Yes

$\Downarrow$

$$\sqrt{a^2 - x^2} \rightarrow x = a \sin\theta$$

$$\sqrt{a^2 + x^2} \rightarrow x = a \tan\theta$$

$$\sqrt{x^2 - a^2} \rightarrow x = a \sec\theta$$

5.3. Common Errors

Students often make the following mistakes.

1. Choosing the wrong substitution.
2. Forgetting to substitute for dx.
3. Forgetting to simplify the radical.
4. Leaving the answer in terms of θ.
5. Using the wrong trigonometric identity.
6. Forgetting to draw a reference triangle.
7. Forgetting absolute values inside logarithms.
8. Using trigonometric substitution when ordinary u-substitution is easier.

5.4. Summary of Important Formulas

Inverse Trigonometric Derivatives

$$\boxed{\frac{d}{dx}\sin^{-1} x = \frac{1}{\sqrt{1-x^2}}}$$

$$\boxed{\frac{d}{dx}\cos^{-1} x = -\frac{1}{\sqrt{1-x^2}}}$$

$$\boxed{\frac{d}{dx}\tan^{-1} x = \frac{1}{1+x^2}}$$

Inverse Trigonometric Integrals

$$\boxed{\int \frac{dx}{\sqrt{1-x^2}} = \sin^{-1} x + C}$$

$$\boxed{\int \frac{dx}{1+x^2} = \tan^{-1} x + C}$$

Trigonometric Substitutions

$\sqrt{a^2 - x^2}$	$x = a\sin\theta$
$\sqrt{a^2 + x^2}$	$x = a\tan\theta$
$\sqrt{x^2 - a^2}$	$x = a\sec\theta$

6. Practice Problems

Evaluate each integral.

1. $$\int \sqrt{9 - x^2}\, dx$$

2. $$\int \frac{dx}{\sqrt{16 - x^2}}$$

3. $$\int \frac{x^2}{\sqrt{x^2 + 9}} dx$$

4. $$\int \frac{x}{\sqrt{x^2 + 25}} dx$$

5. $$\int \frac{dx}{x^2\sqrt{x^2 - 9}}$$

6. $$\int \frac{dx}{9 + x^2}$$

7. $$\int \frac{dx}{\sqrt{25 - x^2}}$$

8. $$\int \frac{x^3}{\sqrt{x^2 + 1}} dx$$

9. $$\int \sqrt{4 - x^2} dx$$

10. $$\int \frac{dx}{\sqrt{x^2 + 36}}$$