Calculus II

Lecture 3

§7.4 Integration by Partial Fractions

Contents

1. Partial Fraction Decomposition and Construction

One of the most useful techniques for integrating rational functions is **partial fraction decomposition**. The main idea is to rewrite a complicated rational function as the sum of several simpler rational functions whose antiderivatives are easy to compute.

Recall that a **rational function** is any function of the form

$$R(x) = \frac{P(x)}{Q(x)},$$

where

$$P(x)$$

and

$$Q(x)$$

are polynomials and

$$Q(x) \neq 0.$$

Examples include

$$\frac{x+1}{x^2-1}, \qquad \frac{2x^2+5x-1}{x^3+x}, \qquad \frac{x^4+3}{x^2+1}.$$

Many rational functions are difficult—or even impossible—to integrate directly. The purpose of partial fraction decomposition is to express the function as a sum of simpler fractions whose denominators contain only one factor.

1.1 Motivation

Consider the rational function

$$\frac{x+5}{x^2+x-2}.$$

At first glance, there is no obvious substitution or elementary integration formula that applies. However, observe that

$$x^2+x-2 = (x+2)(x-1).$$

If we can rewrite

$$\frac{x+5}{(x+2)(x-1)}$$

as

$$\frac{A}{x+2} + \frac{B}{x-1},$$

then each term can be integrated immediately because

$$\int \frac{1}{x-a}\,dx = \ln|x-a| + C.$$

Thus, the difficult integral has been transformed into two elementary logarithmic integrals.
This simple observation is the foundation of the partial fraction method.

1.2 The Goal of Partial Fraction Decomposition

The objective is to rewrite

$$\frac{P(x)}{Q(x)}$$

as

$$\frac{P(x)}{Q(x)} = F_1(x) + F_2(x) + \cdots + F_n(x),$$

where each denominator contains only one factor.
These factors must be one of the following:

1. Linear factors

$$ax+b.$$

2. Repeated linear factors

$$(ax + b)^m.$$

3. Irreducible quadratic factors

$$ax^2 + bx + c,$$

where

$$b^2 - 4ac < 0.$$

4. Repeated irreducible quadratic factors

$$(ax^2 + bx + c)^m.$$

Once the decomposition has been found, each fraction can be integrated individually.

1.3 Proper and Improper Rational Functions

Before attempting a partial fraction decomposition, we must determine whether the rational function is **proper** or **improper**.

Definition A rational function

$$\frac{P(x)}{Q(x)}$$

is called

- **Proper** if

$$\deg(P) < \deg(Q).$$

- **Improper** if

$$\deg(P) \geq \deg(Q).$$

Only proper rational functions may be decomposed directly into partial fractions.

1.4 Examples

$$\frac{x+1}{x^2+4}$$

is proper because

$$1 < 2.$$

$$\frac{x^3+2x+1}{x^2+1}$$

is improper because

$$3 > 2.$$

Polynomial division must be performed before beginning the decomposition.

1.5 Overview of the Partial Fraction Algorithm

Every decomposition follows the same sequence of steps.

1. Determine whether the rational function is proper.
2. If necessary, perform polynomial division.
3. Factor the denominator completely.
4. Write the general partial fraction decomposition.
5. Determine the unknown constants.
6. Integrate each simpler fraction separately.

Every example in this chapter follows this procedure.

Remark. Partial fraction decomposition is an algebraic technique rather than an integration technique.
The decomposition itself does not evaluate the integral.
Instead, it transforms a complicated rational function into simpler pieces whose antiderivatives are already known.
This is why the majority of the work occurs before any integration is performed.

1.6 Polynomial Division (Review)

Before performing a partial fraction decomposition, we must determine whether the rational function is proper or improper.
Recall that a rational function

$$\frac{P(x)}{Q(x)}$$

is improper whenever

$$\deg(P) \geq \deg(Q).$$

Partial fraction decomposition can only be applied directly to proper rational functions. Therefore, whenever the numerator has degree greater than or equal to the degree of the denominator, polynomial long division must first be performed.
The result of the division can always be written in the form

$$\boxed{\frac{P(x)}{Q(x)} = S(x) + \frac{R(x)}{Q(x)},}$$

where

- $S(x)$ is the quotient,
- $R(x)$ is the remainder,
- and

$$\deg(R) < \deg(Q).$$

Only the remainder term

$$\frac{R(x)}{Q(x)}$$

is decomposed into partial fractions.

Why Polynomial Division Comes First Suppose we attempt to decompose

$$\frac{x^3+2x+1}{x^2+1}.$$

Since

$$3>2,$$

the numerator has higher degree than the denominator.
Trying to write

$$\frac{x^3+2x+1}{x^2+1}=\frac{A}{x+i}+\frac{B}{x-i}$$

(or any similar decomposition) immediately fails because the numerator is too large.
Polynomial division removes the excess degree and leaves a proper rational function that can then be decomposed.

1.7 Example: Polynomial Division

Rewrite

$$\frac{x^3+2x+1}{x^2+1}$$

so that the remaining fraction is proper.
Divide

$$x^3+2x+1$$

by

$$x^2+1.$$

The first term of the quotient is

$$x,$$

since

$$x\cdot x^2=x^3.$$

Write

$$x$$

in the quotient.
Subtract

$$x(x^2+1) = x^3 + x$$

from the numerator.

$$x^3 + 2x + 1 - (x^3 + x) = x + 1.$$

Since

$$\deg(x+1) < 2,$$

the division is complete.
Thus,

$$\boxed{\frac{x^3+2x+1}{x^2+1} = x + \frac{x+1}{x^2+1}.}$$

Notice that

$$\frac{x+1}{x^2+1}$$

is now a proper rational function and may be decomposed further if necessary.

1.8 Important Observation

Polynomial division changes neither the function nor the value of the integral.
It merely rewrites the rational function into a more convenient form.
Whenever you encounter an improper rational function, always perform polynomial division before attempting partial fraction decomposition.

1.9 General Procedure

For every rational function, begin with the following checklist.

1. Compare the degrees of the numerator and denominator.
2. If the numerator degree is greater than or equal to the denominator degree, perform polynomial long division.
3. Keep only the proper rational part.
4. Factor the denominator completely.
5. Begin constructing the partial fraction decomposition.

This first step is essential because every decomposition theorem assumes that the rational function is proper.

Remark. Many students make the mistake of trying to factor the denominator before checking whether polynomial division is necessary.
The correct order is always

$$\boxed{\text{Check degrees} \longrightarrow \text{Polynomial Division} \longrightarrow \text{Factor the denominator} \longrightarrow \text{Partial Fraction Decomposition.}}$$

1.10 Obtaining the General Partial Fraction Form

After polynomial division (if necessary), the next step is to factor the denominator completely.

The Fundamental Theorem of Algebra guarantees that every polynomial with real coefficients can be factored into a product of linear factors and irreducible quadratic factors.
Consequently, every proper rational function can be expressed as a sum of partial fractions whose denominators contain only one factor.
The form of each partial fraction depends entirely on the type of factor that appears in the denominator.
The four cases encountered in Calculus II are summarized below.

Case 1: Distinct Linear Factors Suppose the denominator contains a linear factor

$$ax + b.$$

Then the corresponding partial fraction is

$$\boxed{\frac{A}{ax+b}.}$$

Here, A is an unknown constant.

Example If the denominator contains

$$(x-3)(x+2),$$

then the decomposition must include

$$\boxed{\frac{A}{x-3} + \frac{B}{x+2}.}$$

Notice that every distinct linear factor contributes exactly one constant.

Case 2: Repeated Linear Factors Suppose the denominator contains

$$(ax+b)^n,$$

where

$$n \geq 2.$$

A common mistake is to include only one fraction.
Instead, every power of the repeated factor must appear.
For example,

$$(ax+b)^3$$

requires

$$\boxed{\frac{A}{ax+b} + \frac{B}{(ax+b)^2} + \frac{C}{(ax+b)^3}.}$$

In general,

$$\boxed{\frac{A_1}{ax+b} + \frac{A_2}{(ax+b)^2} + \cdots + \frac{A_n}{(ax+b)^n}.}$$

Example For

$$(x+1)^4,$$

the decomposition is

$$\boxed{\frac{A}{x+1}+\frac{B}{(x+1)^2}+\frac{C}{(x+1)^3}+\frac{D}{(x+1)^4}.}$$

Each denominator receives its own unknown constant.

Case 3: Irreducible Quadratic Factors Some quadratic polynomials cannot be factored over the real numbers.
For example,

$$x^2+x+1$$

has discriminant

$$1-4=-3<0,$$

so it has no real roots.
Such a polynomial is called an **irreducible quadratic factor**.
Unlike linear factors, the numerator must now be linear.
Therefore,

$$\boxed{\frac{Ax+B}{ax^2+bx+c}.}$$

The numerator contains two unknown constants because it must have degree strictly less than the denominator.

Example For

$$x^2+4,$$

the corresponding partial fraction is

$$\boxed{\frac{Ax+B}{x^2+4}.}$$

Case 4: Repeated Irreducible Quadratic Factors If the quadratic factor is repeated,

$$(ax^2+bx+c)^n,$$

then every power must appear.
For

$$(ax^2+bx+c)^2,$$

we obtain

$$\boxed{\frac{Ax+B}{ax^2+bx+c} + \frac{Cx+D}{(ax^2+bx+c)^2}.}$$

Similarly,

$$(ax^2+bx+c)^3$$

requires three separate fractions.

1.11 Summary Table

The four cases are summarized below.

Factor	Partial Fraction Form
$ax+b$	$\dfrac{A}{ax+b}$
$(ax+b)^n$	$\dfrac{A_1}{ax+b} + \dfrac{A_2}{(ax+b)^2} + \cdots + \dfrac{A_n}{(ax+b)^n}$
ax^2+bx+c	$\dfrac{Ax+B}{ax^2+bx+c}$
$(ax^2+bx+c)^n$	$\dfrac{A_1x+B_1}{ax^2+bx+c} + \cdots + \dfrac{A_nx+B_n}{(ax^2+bx+c)^n}$

This table should be memorized. Every partial fraction decomposition begins by identifying which of these four cases applies to each factor of the denominator.

1.12 Worked Example: Constructing the General Form

Construct the partial fraction decomposition of

$$\frac{1}{(x+1)^3(x+2)(x^2+x+1)^2}.$$

No constants are determined at this stage.
We only write the correct general form.
The denominator contains

- one repeated linear factor $(x+1)^3$,
- one distinct linear factor $(x+2)$,
- one repeated irreducible quadratic factor $(x^2+x+1)^2$.

Therefore,

$$\boxed{\begin{aligned}\frac{1}{(x+1)^3(x+2)(x^2+x+1)^2} &= \frac{A}{x+2} + \frac{B}{x+1} + \frac{C}{(x+1)^2} + \frac{D}{(x+1)^3} \\ &\quad + \frac{Ex+F}{x^2+x+1} + \frac{Gx+H}{(x^2+x+1)^2}.\end{aligned}}$$

Notice that we have not computed any of the constants.
Our goal in this step is only to write the correct structure of the decomposition.
The unknown constants will be determined in the next section.

Remark. Students often lose points by writing an incorrect general form.
Remember the following rules.

1. Every distinct linear factor receives one constant.
2. Every repeated linear factor receives one fraction for each power.
3. Every irreducible quadratic factor receives a linear numerator.
4. Every repeated irreducible quadratic factor receives one fraction for each power, each with its own linear numerator.

If the general form is incorrect, every subsequent calculation will also be incorrect, even if the algebra is performed perfectly.

1.13 Determining the Constants by Evaluating

Once the correct general form of the partial fraction decomposition has been constructed, the next task is to determine the unknown constants.

Several methods are available for finding these constants. The simplest and most efficient method, whenever applicable, is the **evaluation method**, sometimes called the *cover-up method*.

The main idea is to choose values of the variable that cause most of the terms to vanish, leaving only one unknown constant.

General Procedure Suppose

$$\frac{P(x)}{Q(x)} = \frac{A}{x-a} + \frac{B}{x-b}.$$

To determine the constants, proceed as follows.

1. Multiply both sides by the common denominator.
2. Substitute values of x that make one or more factors equal to zero.
3. Solve for the remaining unknown constant.
4. Repeat the process until every constant has been determined.

Whenever the denominator factors into distinct linear factors, this method is usually much faster than solving a system of equations.

1.14 Why the Evaluation Method Works

Suppose

$$\frac{P(x)}{(x-a)(x-b)} = \frac{A}{x-a} + \frac{B}{x-b}.$$

Multiplying both sides by

$$(x-a)(x-b)$$

gives

$$P(x) = A(x-b) + B(x-a).$$

Now evaluate the equation at

$$x = a.$$

Since

$$x - a = 0,$$

the entire term containing

$$B$$

vanishes.
Thus,

$$P(a) = A(a - b).$$

The constant

$$A$$

can now be found immediately.
Similarly,
setting

$$x = b$$

eliminates the term containing

$$A,$$

allowing us to determine

$$B.$$

This explains why the evaluation method is so effective.

1.15 Worked Example

Determine the partial fraction decomposition of

$$\frac{12x - 10}{8x^2 + 10x - 3}.$$

Step 1. Factor the Denominator Begin by factoring

$$8x^2 + 10x - 3.$$

We obtain

$$8x^2 + 10x - 3 = (4x - 1)(2x + 3).$$

Therefore,

$$\frac{12x - 10}{8x^2 + 10x - 3} = \frac{A}{4x - 1} + \frac{B}{2x + 3}.$$

Step 2. Multiply Through Multiply both sides by

$$(4x - 1)(2x + 3).$$

This gives

$$12x - 10 = A(2x + 3) + B(4x - 1).$$

Notice that the denominators have disappeared.

Step 3. Determine A Choose

$$x = \frac{1}{4},$$

since this makes

$$4x - 1 = 0.$$

Substituting,

$$12\left(\frac{1}{4}\right) - 10 = A\left(2 \cdot \frac{1}{4} + 3\right).$$

Compute each side.
The left-hand side is

$$3 - 10 = -7.$$

The right-hand side is

$$A\left(\frac{1}{2} + 3\right) = A\left(\frac{7}{2}\right).$$

Hence,

$$-7 = \frac{7}{2}A.$$

Multiply both sides by

$$\frac{2}{7}.$$

Therefore,

$$\boxed{A = -2.}$$

Step 4. Determine B Now choose

$$x = -\frac{3}{2},$$

which makes

$$2x + 3 = 0.$$

Substituting,

$$12\left(-\frac{3}{2}\right) - 10 = B\left(4\left(-\frac{3}{2}\right) - 1\right).$$

Simplify.
The left-hand side is

$$-18 - 10 = -28.$$

The right-hand side becomes

$$B(-6-1) = -7B.$$

Hence,

$$-28 = -7B.$$

Dividing by

$$-7$$

gives

$$\boxed{B = 4.}$$

Step 5. Write the Final Decomposition Substituting the constants,

$$\boxed{\frac{12x-10}{8x^2+10x-3} = -\frac{2}{4x-1} + \frac{4}{2x+3}.}$$

Verification. Whenever possible, verify the decomposition by combining the fractions.
Indeed,

$$\begin{aligned} -\frac{2}{4x-1} + \frac{4}{2x+3} &= \frac{-2(2x+3)+4(4x-1)}{(4x-1)(2x+3)} \\ &= \frac{-4x-6+16x-4}{(4x-1)(2x+3)} \\ &= \frac{12x-10}{8x^2+10x-3}, \end{aligned}$$

which agrees with the original rational function.

Advantages of the Evaluation Method. The evaluation method has several advantages.

- It avoids solving a system of linear equations.
- Each substitution often determines one constant immediately.
- The computations are usually shorter than the method of equating coefficients.
- It works particularly well when the denominator contains distinct linear factors.

Limitations. The evaluation method is not always sufficient.
For repeated linear factors or irreducible quadratic factors, evaluating at special values generally does not determine every constant.
In such situations, we must supplement this method with the technique of equating coefficients, which is discussed in the next section.

1.16 Determining the Constants by Equating Coefficients

The evaluation method is extremely efficient when the denominator contains distinct linear factors. However, it is not always possible to eliminate all unknown constants by substituting convenient values of the variable.

For repeated factors and irreducible quadratic factors, a more general approach is required. This approach is known as the **method of equating coefficients**.
The basic idea is simple. Since two polynomials are equal if and only if the coefficients of corresponding powers of x are equal, we may compare the coefficients on both sides of the identity to obtain a system of linear equations.

General Procedure To determine the unknown constants,

1. Write the general partial fraction decomposition.
2. Multiply both sides by the common denominator.
3. Expand the right-hand side.
4. Collect like powers of x.
5. Equate the coefficients of corresponding powers of x.
6. Solve the resulting system of linear equations.

Unlike the evaluation method, this procedure always works, although it may require more algebra.

1.17 Worked Example

Determine the partial fraction decomposition of

$$\frac{4x^3 + x^2 + 3x - 3}{(x^2 + x + 1)^2}.$$

Step 1. Write the General Form Since the denominator contains the repeated irreducible quadratic factor

$$(x^2 + x + 1)^2,$$

the decomposition must be

$$\boxed{\frac{4x^3 + x^2 + 3x - 3}{(x^2 + x + 1)^2} = \frac{Ax + B}{x^2 + x + 1} + \frac{Cx + D}{(x^2 + x + 1)^2}.}$$

Notice that each numerator is linear because the denominator is an irreducible quadratic polynomial.

Step 2. Clear the Denominator Multiply both sides by

$$(x^2 + x + 1)^2.$$

Then

$$4x^3 + x^2 + 3x - 3 = (Ax + B)(x^2 + x + 1) + (Cx + D).$$

Step 3. Expand the Product Expand

$$(Ax + B)(x^2 + x + 1).$$

First distribute Ax:

$$Ax(x^2 + x + 1) = Ax^3 + Ax^2 + Ax.$$

Next distribute B:

$$B(x^2 + x + 1) = Bx^2 + Bx + B.$$

Adding these terms,

$$(Ax + B)(x^2 + x + 1) = Ax^3 + (A + B)x^2 + (A + B)x + B.$$

Finally include the remaining term

$$Cx + D.$$

Therefore,

$$\boxed{4x^3 + x^2 + 3x - 3 = Ax^3 + (A + B)x^2 + (A + B + C)x + (B + D).}$$

Step 4. Compare Coefficients Since these two polynomials are identical, their coefficients must agree.
Matching powers of x,

$$\begin{aligned} x^3 &: \quad A = 4, \\ x^2 &: \quad A + B = 1, \\ x^1 &: \quad A + B + C = 3, \\ x^0 &: \quad B + D = -3. \end{aligned}$$

Thus we obtain the linear system

$$\boxed{\begin{aligned} A &= 4, \\ A + B &= 1, \\ A + B + C &= 3, \\ B + D &= -3. \end{aligned}}$$

Step 5. Solve the System The first equation immediately gives

$$A = 4.$$

Substituting into the second equation,

$$4 + B = 1,$$

so

$$B = -3.$$

Now substitute A and B into the third equation,

$$4 - 3 + C = 3,$$

which yields

$$C = 2.$$

Finally,

$$-3 + D = -3,$$

so

$$D = 0.$$

Therefore,

$$\boxed{A = 4, \qquad B = -3, \qquad C = 2, \qquad D = 0.}$$

Step 6. Write the Final Decomposition Substituting the constants,

$$\boxed{\frac{4x^3 + x^2 + 3x - 3}{(x^2 + x + 1)^2} = \frac{4x - 3}{x^2 + x + 1} + \frac{2x}{(x^2 + x + 1)^2}.}$$

Verification. A useful way to verify the result is to substitute the decomposition back into a common denominator and simplify.
Since

$$(4x - 3)(x^2 + x + 1) + 2x = 4x^3 + x^2 + 3x - 3,$$

the decomposition agrees with the original rational function.

1.18 Comparison of the Two Methods

There are two standard methods for determining unknown constants.

Method	**Best Used When**
$Evaluation$	$Distinctlinearfactors$
$Equatingcoefficients$	$Repeatedfactorsorirreduciblequadratics$

In practice, many problems are solved by combining both methods. First, evaluate at convenient values of x to determine as many constants as possible. Then use equating coefficients to determine the remaining unknowns.

Remark. The most common mistake is expanding the product incorrectly. When solving these problems, it is worth taking a few extra moments to carefully distribute each term and collect like powers of x. A small algebraic error at this stage leads to an incorrect system of equations and, consequently, an incorrect decomposition.

2. Integration by Partial Fractions

In the previous section, we learned how to decompose a rational function into simpler rational functions.
We now study the main purpose of this decomposition.
Our goal is to evaluate integrals of rational functions.
Recall that a rational function has the form

$$R(x) = \frac{P(x)}{Q(x)},$$

where $P(x)$ and $Q(x)$ are polynomials.

Although many rational functions appear complicated, every proper rational function can be rewritten as the sum of simpler fractions whose antiderivatives are already known.
Consequently, partial fraction decomposition transforms a difficult integral into several elementary integrals.

2.1 The Basic Idea

Suppose we wish to compute

$$\int \frac{P(x)}{Q(x)}\,dx.$$

The integration process always follows the same sequence of steps.

1. If necessary, perform polynomial division.
2. Factor the denominator completely.
3. Construct the correct partial fraction decomposition.
4. Determine all unknown constants.
5. Integrate each partial fraction separately.

Instead of attempting to integrate one complicated fraction, we integrate several much simpler fractions.

2.2 Why the Method Works

Suppose

$$\frac{P(x)}{Q(x)} = F_1(x) + F_2(x) + \cdots + F_n(x).$$

Since integration is a linear operation,

$$\boxed{\int \frac{P(x)}{Q(x)}\,dx = \int F_1(x)\,dx + \int F_2(x)\,dx + \cdots + \int F_n(x)\,dx.}$$

Thus, after decomposition, each integral can be evaluated independently.

2.3 General Algorithm

Whenever you encounter a rational function, follow the algorithm below.

1. Check whether the rational function is proper.

   If not, perform polynomial division.
2. Factor the denominator completely.

   Factor into

   - linear factors,
   - repeated linear factors,
   - irreducible quadratic factors,
   - repeated irreducible quadratic factors.
3. Write the general partial fraction decomposition.
4. Determine the unknown constants.

   Use

- evaluation,
- equating coefficients,
- or a combination of both.

5. Integrate every resulting fraction separately.

This algorithm applies to every rational function encountered in Calculus II.

2.4 Common Elementary Integrals

After decomposition, nearly every integral reduces to one of a few standard forms.

Linear Denominator

$$\boxed{\int \frac{dx}{ax+b} = \frac{1}{a}\ln|ax+b| + C.}$$

Repeated Linear Factor For

$$n > 1,$$

$$\boxed{\int \frac{dx}{(ax+b)^n} = -\frac{1}{a(n-1)}(ax+b)^{-(n-1)} + C.}$$

Irreducible Quadratic Frequently,

$$\frac{Ax+B}{x^2+bx+c}$$

is separated into

$$A\frac{2x+b}{x^2+bx+c} + \frac{\text{constant}}{x^2+bx+c},$$

so that one integral becomes logarithmic while the other becomes an inverse tangent integral after completing the square.

Repeated Quadratics Repeated quadratic factors are usually handled by combining substitution and standard inverse trigonometric integrals.

2.5 A Flowchart for Rational Integrals

The following diagram summarizes the entire procedure.

$$\boxed{\begin{array}{c}\text{Rational Function}\\ \downarrow\\ \text{Proper?}\\ \downarrow\\ \text{Polynomial Division (if necessary)}\\ \downarrow\\ \text{Factor Denominator}\\ \downarrow\\ \text{Partial Fraction Decomposition}\\ \downarrow\\ \text{Determine Constants}\\ \downarrow\\ \text{Integrate Each Fraction}\end{array}}$$

Remark. The integration itself is usually the easiest part of the problem.
The majority of the work lies in obtaining the correct partial fraction decomposition.
Once the decomposition has been completed correctly, the resulting integrals are almost always standard logarithmic, inverse trigonometric, or elementary power-rule integrals.

2.6 Example 1: A Rational Function with Distinct Linear Factors

Evaluate

$$I = \int \frac{x+5}{x^2+x-2}\,dx.$$

Step 1. Verify that the Rational Function is Proper Before applying partial fractions, compare the degrees of the numerator and the denominator.
The numerator has degree

$$1,$$

while the denominator has degree

$$2.$$

Since

$$1 < 2,$$

the rational function is already proper.
Therefore, polynomial division is not necessary.

Step 2. Factor the Denominator Factor

$$x^2+x-2.$$

We seek two numbers whose product is

$$-2$$

and whose sum is

$$1.$$

These numbers are

$$2 \quad \text{and} \quad -1.$$

Hence,

$$x^2 + x - 2 = (x+2)(x-1).$$

The integral now becomes

$$I = \int \frac{x+5}{(x+2)(x-1)}\,dx.$$

Step 3. Construct the Partial Fraction Decomposition Since the denominator contains two distinct linear factors, we write

$$\boxed{\frac{x+5}{(x+2)(x-1)} = \frac{A}{x+2} + \frac{B}{x-1}.}$$

The constants A and B remain to be determined.

Step 4. Determine the Constants Multiply both sides by

$$(x+2)(x-1).$$

This eliminates the denominators and produces

$$x + 5 = A(x-1) + B(x+2).$$

Finding A Choose

$$x = -2,$$

because

$$x + 2 = 0.$$

Substituting,

$$-2 + 5 = A(-3).$$

Therefore,

$$3 = -3A,$$

which gives

$$\boxed{A = -1.}$$

Finding B Choose

$$x = 1,$$

because

$$x - 1 = 0.$$

Substituting,

$$1 + 5 = B(3).$$

Thus,

$$6 = 3B,$$

and

$$\boxed{B = 2.}$$

Step 5. Write the Decomposition Substituting the constants,

$$\boxed{\frac{x+5}{x^2+x-2} = -\frac{1}{x+2} + \frac{2}{x-1}.}$$

Step 6. Integrate Each Term Using the linearity of integration,

$$I = \int \left(-\frac{1}{x+2} + \frac{2}{x-1} \right) dx.$$

Separate the integral,

$$I = -\int \frac{dx}{x+2} + 2\int \frac{dx}{x-1}.$$

Recall that

$$\int \frac{dx}{x-a} = \ln|x-a| + C.$$

Therefore,

$$I = -\ln|x+2| + 2\ln|x-1| + C.$$

Hence,

$$\boxed{\int \frac{x+5}{x^2+x-2}\, dx = -\ln|x+2| + 2\ln|x-1| + C.}$$

Verification Differentiate the answer.

$$\frac{d}{dx}\left(-\ln|x+2|+2\ln|x-1|\right)=-\frac{1}{x+2}+\frac{2}{x-1}.$$

Combining these fractions,

$$\begin{aligned}-\frac{1}{x+2}+\frac{2}{x-1}&=\frac{-(x-1)+2(x+2)}{(x+2)(x-1)}\\&=\frac{x+5}{x^2+x-2},\end{aligned}$$

which is exactly the original integrand.
Therefore, the solution is correct.

Discussion. This example illustrates the standard procedure for integrating rational functions whose denominator factors into distinct linear factors.
The important steps were:

1. Verify that the fraction is proper.
2. Factor the denominator.
3. Construct the partial fraction decomposition.
4. Determine the constants.
5. Integrate each elementary fraction separately.

Most introductory partial fraction problems follow this pattern.

2.7 Example 2: Polynomial Division and a Repeated Linear Factor

Evaluate

$$I=\int\frac{x^4-2x^2+4x+1}{x^3-x^2-x+1}\,dx.$$

This example is more involved than the previous one for two reasons:

1. the rational function is improper, so polynomial division is required;
2. the denominator contains a repeated linear factor.

Step 1. Check the Degrees The numerator has degree 4, while the denominator has degree 3. Since

$$4\geq 3,$$

the rational function is improper. Therefore, we must perform polynomial division before using partial fractions.

Step 2. Perform Polynomial Division Divide

$$x^4-2x^2+4x+1$$

by

$$x^3-x^2-x+1.$$

For clarity, include the missing x^3-term in the numerator:

$$x^4 + 0x^3 - 2x^2 + 4x + 1.$$

The leading-term division gives

$$\frac{x^4}{x^3} = x.$$

Multiplying the denominator by x,

$$x(x^3 - x^2 - x + 1) = x^4 - x^3 - x^2 + x.$$

Subtract:

$$\begin{aligned}&(x^4 + 0x^3 - 2x^2 + 4x + 1) - (x^4 - x^3 - x^2 + x)\\ &\quad = x^3 - x^2 + 3x + 1.\end{aligned}$$

Now divide the leading terms again:

$$\frac{x^3}{x^3} = 1.$$

Multiply the denominator by 1:

$$x^3 - x^2 - x + 1.$$

Subtract:

$$\begin{aligned}&(x^3 - x^2 + 3x + 1) - (x^3 - x^2 - x + 1)\\ &\quad = 4x.\end{aligned}$$

Therefore,

$$\boxed{\frac{x^4 - 2x^2 + 4x + 1}{x^3 - x^2 - x + 1} = x + 1 + \frac{4x}{x^3 - x^2 - x + 1}.}$$

The remainder has degree 1, which is less than the degree 3 of the denominator, so the remaining rational function is proper.

Step 3. Factor the Denominator Factor by grouping:

$$\begin{aligned}x^3 - x^2 - x + 1 &= x^2(x-1) - 1(x-1)\\ &= (x^2 - 1)(x - 1)\\ &= (x+1)(x-1)^2.\end{aligned}$$

Hence,

$$I = \int \left[x + 1 + \frac{4x}{(x+1)(x-1)^2}\right] dx.$$

Step 4. Write the Partial Fraction Form The denominator contains

- the distinct linear factor $x + 1$;

- the repeated linear factor $(x-1)^2$.

Therefore,

$$\boxed{\frac{4x}{(x+1)(x-1)^2} = \frac{A}{x+1} + \frac{B}{x-1} + \frac{C}{(x-1)^2}.}$$

It is essential to include fractions corresponding to both powers

$$x-1 \qquad \text{and} \qquad (x-1)^2.$$

Step 5. Clear the Denominators Multiply both sides by

$$(x+1)(x-1)^2.$$

This gives

$$\boxed{4x = A(x-1)^2 + B(x+1)(x-1) + C(x+1).}$$

Step 6. Determine A Let

$$x = -1.$$

Then the terms containing B and C vanish:

$$4(-1) = A(-2)^2.$$

Thus,

$$-4 = 4A,$$

so

$$\boxed{A = -1.}$$

Step 7. Determine C Let

$$x = 1.$$

Then the terms containing A and B vanish:

$$4(1) = C(2).$$

Therefore,

$$4 = 2C,$$

and hence

$$\boxed{C = 2.}$$

Step 8. Determine B The special values $x=-1$ and $x=1$ determine A and C, but not B. Choose a convenient additional value, such as

$$x = 0.$$

Substituting into

$$4x = A(x-1)^2 + B(x+1)(x-1) + C(x+1),$$

gives

$$0 = A - B + C.$$

Using

$$A = -1, \qquad C = 2,$$

we obtain

$$0 = -1 - B + 2.$$

Therefore,

$$B = 1.$$

Thus,

$$\boxed{B = 1.}$$

Step 9. Write the Complete Decomposition Substituting the constants,

$$\frac{4x}{(x+1)(x-1)^2} = -\frac{1}{x+1} + \frac{1}{x-1} + \frac{2}{(x-1)^2}.$$

Therefore,

$$\boxed{\frac{x^4 - 2x^2 + 4x + 1}{x^3 - x^2 - x + 1} = x + 1 - \frac{1}{x+1} + \frac{1}{x-1} + \frac{2}{(x-1)^2}.}$$

Step 10. Integrate Term by Term Now write

$$I = \int \left(x + 1 - \frac{1}{x+1} + \frac{1}{x-1} + \frac{2}{(x-1)^2} \right) dx.$$

Integrate each term separately:

$$\int x\,dx = \frac{x^2}{2}, \qquad \int 1\,dx = x,$$

$$\int -\frac{1}{x+1}\,dx = -\ln|x+1|,$$

$$\int \frac{1}{x-1}\,dx = \ln|x-1|.$$

For the repeated-factor term,

$$\int \frac{2}{(x-1)^2}\,dx = 2\int (x-1)^{-2}\,dx.$$

Using the power rule,

$$2\int (x-1)^{-2}\,dx = -2(x-1)^{-1}.$$

Hence,

$$\boxed{\int \frac{x^4-2x^2+4x+1}{x^3-x^2-x+1}\,dx = \frac{x^2}{2} + x - \ln|x+1| + \ln|x-1| - \frac{2}{x-1} + C.}$$

Verification of the Decomposition Combine the partial fractions:

$$-\frac{1}{x+1} + \frac{1}{x-1} + \frac{2}{(x-1)^2} = \frac{-(x-1)^2 + (x+1)(x-1) + 2(x+1)}{(x+1)(x-1)^2}.$$

Simplifying the numerator,

$$\begin{aligned} -(x-1)^2 + (x+1)(x-1) + 2(x+1) &= -(x^2-2x+1) + (x^2-1) + 2x + 2 \\ &= 4x. \end{aligned}$$

Therefore,

$$-\frac{1}{x+1} + \frac{1}{x-1} + \frac{2}{(x-1)^2} = \frac{4x}{(x+1)(x-1)^2},$$

which confirms the decomposition.

Important Lessons from Example 2. This example illustrates several essential features of the method:

1. Always perform polynomial division when the rational function is improper.
2. Factor the denominator completely before writing the decomposition.
3. A repeated linear factor $(x-a)^2$ requires both

$$\frac{A}{x-a} \qquad \text{and} \qquad \frac{B}{(x-a)^2}.$$

4. Evaluation at roots may determine some, but not necessarily all, of the constants.
5. Repeated linear factors produce power-rule terms in addition to logarithmic terms.

2.8 Example 3: A Linear Factor and a Repeated Irreducible Quadratic

Evaluate

$$I = \int \frac{1-x+2x^2-x^3}{x(x^2+1)^2}\,dx.$$

This example introduces a denominator containing both

$$x,$$

which is a linear factor, and

$$(x^2+1)^2,$$

which is a repeated irreducible quadratic factor.
The decomposition will therefore contain both constant and linear numerators.

Step 1. Verify that the Rational Function is Proper The numerator has degree 3, while the denominator has degree

$$1+2(2)=5.$$

Since

$$3<5,$$

the rational function is proper, so polynomial division is not required.

Step 2. Identify the Denominator Factors The denominator is already factored:

$$x(x^2+1)^2.$$

The factor x is linear.
The quadratic factor

$$x^2+1$$

is irreducible over the real numbers because its discriminant is

$$0^2-4(1)(1)=-4<0.$$

Since x^2+1 appears with multiplicity 2, the decomposition must include a fraction for each power:

$$x^2+1 \qquad \text{and} \qquad (x^2+1)^2.$$

Step 3. Write the General Partial Fraction Form The linear factor x contributes

$$\frac{A}{x}.$$

The repeated irreducible quadratic factor contributes

$$\frac{Bx+C}{x^2+1}+\frac{Dx+E}{(x^2+1)^2}.$$

Therefore,

$$\boxed{\frac{1-x+2x^2-x^3}{x(x^2+1)^2}=\frac{A}{x}+\frac{Bx+C}{x^2+1}+\frac{Dx+E}{(x^2+1)^2}.}$$

Step 4. Clear the Denominators Multiply both sides by

$$x(x^2+1)^2.$$

This gives

$$\boxed{1 - x + 2x^2 - x^3 = A(x^2+1)^2 + (Bx+C)x(x^2+1) + (Dx+E)x.}$$

Step 5. Expand Each Term First,

$$A(x^2+1)^2 = A(x^4+2x^2+1) = Ax^4 + 2Ax^2 + A.$$

Next,

$$\begin{aligned}(Bx+C)x(x^2+1) &= (Bx+C)(x^3+x)\\ &= Bx^4 + Cx^3 + Bx^2 + Cx.\end{aligned}$$

Finally,

$$(Dx+E)x = Dx^2 + Ex.$$

Adding all terms,

$$\begin{aligned}1 - x + 2x^2 - x^3 = {} & (A+B)x^4 + Cx^3\\ & + (2A+B+D)x^2 + (C+E)x + A.\end{aligned}$$

Step 6. Equate Coefficients Write the left-hand polynomial in descending powers:

$$0x^4 - x^3 + 2x^2 - x + 1.$$

Comparing corresponding coefficients gives

$$\boxed{\begin{aligned}A+B &= 0,\\ C &= -1,\\ 2A+B+D &= 2,\\ C+E &= -1,\\ A &= 1.\end{aligned}}$$

Step 7. Solve the System From the constant term,

$$A = 1.$$

Then

$$A + B = 0$$

gives

$$1 + B = 0,$$

so

$$B = -1.$$

We already have

$$C = -1.$$

Next,

$$2A + B + D = 2$$

becomes

$$2(1) - 1 + D = 2,$$

so

$$D = 1.$$

Finally,

$$C + E = -1$$

becomes

$$-1 + E = -1,$$

and therefore

$$E = 0.$$

Thus,

$$\boxed{A = 1, \qquad B = -1, \qquad C = -1, \qquad D = 1, \qquad E = 0.}$$

Step 8. Write the Decomposition Substituting the constants,

$$\boxed{\frac{1 - x + 2x^2 - x^3}{x(x^2+1)^2} = \frac{1}{x} + \frac{-x-1}{x^2+1} + \frac{x}{(x^2+1)^2}.}$$

It is helpful to separate the numerator of the middle fraction:

$$\frac{-x-1}{x^2+1} = -\frac{x}{x^2+1} - \frac{1}{x^2+1}.$$

Therefore,

$$I = \int \left[\frac{1}{x} - \frac{x}{x^2+1} - \frac{1}{x^2+1} + \frac{x}{(x^2+1)^2}\right] dx.$$

Step 9. Integrate the First Term

$$\int \frac{1}{x}\, dx = \ln|x|.$$

Step 10. Integrate the Logarithmic Quadratic Term Consider

$$-\int \frac{x}{x^2+1}\, dx.$$

Let

$$u = x^2 + 1, \qquad du = 2x\,dx.$$

Then

$$x\,dx = \frac{1}{2}\,du.$$

Thus,

$$\begin{aligned} -\int \frac{x}{x^2+1}\,dx &= -\frac{1}{2}\int \frac{du}{u} \\ &= -\frac{1}{2}\ln|u| \\ &= -\frac{1}{2}\ln(x^2+1). \end{aligned}$$

Since

$$x^2 + 1 > 0$$

for every real x, absolute values are unnecessary.

Step 11. Integrate the Inverse Tangent Term Using the standard formula

$$\int \frac{dx}{1+x^2} = \tan^{-1} x + C,$$

we obtain

$$-\int \frac{dx}{x^2+1} = -\tan^{-1} x.$$

Step 12. Integrate the Repeated Quadratic Term Consider

$$\int \frac{x}{(x^2+1)^2}\,dx.$$

Let

$$u = x^2 + 1, \qquad du = 2x\,dx.$$

Then

$$\begin{aligned} \int \frac{x}{(x^2+1)^2}\,dx &= \frac{1}{2}\int u^{-2}\,du \\ &= \frac{1}{2}\left(-u^{-1}\right) \\ &= -\frac{1}{2u} \\ &= -\frac{1}{2(x^2+1)}. \end{aligned}$$

Step 13. Combine the Results Therefore,

$$\boxed{\begin{aligned}\int \frac{1-x+2x^2-x^3}{x(x^2+1)^2}\,dx = {} & \ln|x| - \frac{1}{2}\ln(x^2+1) \\ & - \tan^{-1} x - \frac{1}{2(x^2+1)} + C.\end{aligned}}$$

Verification of the Decomposition Starting with

$$\frac{1}{x} + \frac{-x-1}{x^2+1} + \frac{x}{(x^2+1)^2},$$

the common denominator is

$$x(x^2+1)^2.$$

The numerator becomes

$$(x^2+1)^2 + (-x-1)x(x^2+1) + x^2.$$

Expanding,

$$\begin{aligned}(x^2+1)^2 &= x^4 + 2x^2 + 1, \\ (-x-1)x(x^2+1) &= -x^4 - x^3 - x^2 - x, \\ x^2 &= x^2.\end{aligned}$$

Adding,

$$x^4 + 2x^2 + 1 - x^4 - x^3 - x^2 - x + x^2 = 1 - x + 2x^2 - x^3.$$

Thus the decomposition is correct.

Important Lessons from Example 3. This example illustrates several important principles.

1. An irreducible quadratic factor requires a linear numerator.
2. If an irreducible quadratic factor is repeated, every power must appear in the decomposition.
3. Equating coefficients is often the most efficient method when irreducible quadratic factors are present.
4. A linear numerator over a quadratic denominator frequently splits into two types of integrals:

   a logarithmic integral and an inverse tangent integral.

5. Repeated quadratic factors may produce rational terms such as

$$\frac{1}{x^2+1}$$

   in the final antiderivative.

2.9 Example 4: Substitution Followed by Partial Fractions

Evaluate

$$I = \int \frac{\sqrt{x+4}}{x}\,dx.$$

At first glance, the integrand is not a rational function because it contains the radical

$$\sqrt{x+4}.$$

Therefore, partial fraction decomposition cannot be applied immediately.
The first step is to use a substitution that removes the radical and converts the integral into a rational function.

Step 1. Choose a Substitution Let

$$u = \sqrt{x+4}.$$

Equivalently,

$$u^2 = x + 4.$$

Solving for x,

$$\boxed{x = u^2 - 4.}$$

Differentiate both sides:

$$2u\,du = dx.$$

Thus,

$$\boxed{dx = 2u\,du.}$$

Also,

$$\sqrt{x+4} = u.$$

Step 2. Rewrite the Integral Substitute

$$\sqrt{x+4} = u, \qquad x = u^2 - 4, \qquad dx = 2u\,du.$$

Then

$$\begin{aligned} I &= \int \frac{u}{u^2-4}(2u)\,du \\ &= \int \frac{2u^2}{u^2-4}\,du. \end{aligned}$$

The integral is now rational in u.

Step 3. Perform Algebraic Division The numerator and denominator have the same degree, so the rational function is improper.
Rewrite the numerator as

$$2u^2 = 2(u^2-4) + 8.$$

Therefore,

$$\frac{2u^2}{u^2-4} = 2 + \frac{8}{u^2-4}.$$

Hence,

$$I = \int \left(2 + \frac{8}{u^2-4}\right) du.$$

Step 4. Factor the Denominator Use the difference-of-squares identity:

$$u^2 - 4 = (u-2)(u+2).$$

Thus,

$$\frac{8}{u^2-4} = \frac{8}{(u-2)(u+2)}.$$

Step 5. Write the Partial Fraction Decomposition Because the denominator contains two distinct linear factors, write

$$\frac{8}{(u-2)(u+2)} = \frac{A}{u-2} + \frac{B}{u+2}.$$

Multiply both sides by

$$(u-2)(u+2).$$

Then

$$8 = A(u+2) + B(u-2).$$

Determine A. Set

$$u = 2.$$

Then

$$8 = A(4),$$

so

$$\boxed{A = 2.}$$

Determine B. Set

$$u = -2.$$

Then

$$8 = B(-4),$$

so

$$\boxed{B = -2.}$$

Therefore,

$$\boxed{\frac{8}{u^2-4} = \frac{2}{u-2} - \frac{2}{u+2}.}$$

Step 6. Integrate The integral becomes

$$I = \int \left(2 + \frac{2}{u-2} - \frac{2}{u+2}\right) du.$$

Integrating term by term,

$$\int 2\,du = 2u,$$

$$\int \frac{2}{u-2}\,du = 2\ln|u-2|,$$

and

$$\int -\frac{2}{u+2}\,du = -2\ln|u+2|.$$

Hence,

$$\boxed{I = 2u + 2\ln|u-2| - 2\ln|u+2| + C.}$$

Step 7. Substitute Back Recall that

$$u = \sqrt{x+4}.$$

Therefore,

$$\boxed{\begin{aligned}\int \frac{\sqrt{x+4}}{x}\,dx = 2\sqrt{x+4} + 2\ln\left|\sqrt{x+4}-2\right| \\ - 2\ln\left|\sqrt{x+4}+2\right| + C.\end{aligned}}$$

The logarithmic terms may also be combined:

$$\boxed{\int \frac{\sqrt{x+4}}{x}\,dx = 2\sqrt{x+4} + 2\ln\left|\frac{\sqrt{x+4}-2}{\sqrt{x+4}+2}\right| + C.}$$

Verification Let

$$u = \sqrt{x+4}.$$

Then

$$\frac{du}{dx} = \frac{1}{2\sqrt{x+4}} = \frac{1}{2u}.$$

Differentiate

$$F(x) = 2u + 2\ln|u-2| - 2\ln|u+2|.$$

Using the Chain Rule,

$$\begin{aligned} F'(x) &= 2u' + \frac{2u'}{u-2} - \frac{2u'}{u+2} \\ &= 2u'\left(1 + \frac{1}{u-2} - \frac{1}{u+2}\right). \end{aligned}$$

Since

$$\frac{1}{u-2} - \frac{1}{u+2} = \frac{4}{u^2-4},$$

we obtain

$$F'(x) = 2u'\left(1 + \frac{4}{u^2-4}\right).$$

Using

$$u' = \frac{1}{2u},$$

gives

$$F'(x) = \frac{1}{u}\left(\frac{u^2}{u^2-4}\right) = \frac{u}{u^2-4}.$$

Finally,

$$u = \sqrt{x+4} \qquad \text{and} \qquad u^2 - 4 = x,$$

so

$$F'(x) = \frac{\sqrt{x+4}}{x},$$

which is the original integrand.

Important Lessons from Example 4. This example illustrates that an integral does not need to begin as a rational function in order for partial fractions to be useful.
The sequence of techniques was

$$\boxed{\text{Radical substitution} \longrightarrow \text{Rational function} \longrightarrow \text{Polynomial division} \longrightarrow \text{Partial fractions.}}$$

The main lessons are:

1. When a radical prevents the integrand from being rational, first look for a substitution that removes the radical.
2. After substitution, check whether the resulting rational function is proper.
3. Perform algebraic division before decomposition whenever the numerator degree is greater than or equal to the denominator degree.

4. Partial fraction decomposition may be used together with other integration techniques.
5. After integrating, always return to the original variable.

3. Indeterminate Forms of Limits

In many problems involving limits, direct substitution immediately determines the answer. For example,

$$\lim_{x\to 2}(3x+1) = 7,$$

because polynomial functions are continuous.

However, not every limit can be evaluated by direct substitution. In many cases, substitution produces an expression that provides no information about the actual value of the limit. Such expressions are called **indeterminate forms**.

Indeterminate forms arise frequently in Calculus when studying limits, derivatives, infinite series, and improper integrals. In this section, we review the most common indeterminate forms and introduce one of the most important techniques for evaluating such limits: *L'Hôpital's Rule*.

3.1 Motivation

Consider the limit

$$\lim_{x\to 1}\frac{\ln x}{x-1}.$$

Direct substitution gives

$$\frac{\ln(1)}{1-1} = \frac{0}{0},$$

which is undefined.

Notice that obtaining

$$\frac{0}{0}$$

does *not* mean the limit does not exist. Instead, it simply means that the expression is inconclusive and further analysis is required.

For example,

$$\lim_{x\to 0}\frac{\sin x}{x} = 1,$$

even though direct substitution also produces

$$\frac{0}{0}.$$

Similarly,

$$\lim_{x\to 0}\frac{x^2}{x} = 0,$$

and

$$\lim_{x\to 0}\frac{x}{x^2} = \infty.$$

All three limits initially produce the same indeterminate expression

$$\frac{0}{0},$$

yet their values are completely different.
This demonstrates why the expression

$$\frac{0}{0}$$

is called an **indeterminate form**: it does not determine the value of the limit.

3.2 What Does "Indeterminate" Mean?

An expression is called **indeterminate** if it is impossible to determine the value of a limit solely from the limiting behavior of its numerator and denominator (or factors).
For example,

$$\lim_{x \to a} f(x) = 0, \qquad \lim_{x \to a} g(x) = 0,$$

does not imply

$$\lim_{x \to a} \frac{f(x)}{g(x)}$$

must equal 0, 1, or ∞.
Instead, additional analysis is necessary.

3.3 The Seven Indeterminate Forms

The seven classical indeterminate forms encountered in Calculus are

$$\boxed{\frac{0}{0}}$$

$$\boxed{\frac{\infty}{\infty}}$$

$$\boxed{0 \cdot \infty}$$

$$\boxed{\infty - \infty}$$

$$\boxed{0^0}$$

$$\boxed{\infty^0}$$

$$\boxed{1^\infty}$$

These are called indeterminate because different functions producing the same symbolic form may have completely different limits. :contentReference[oaicite:1]index=1

3.4 Examples of Indeterminate Forms

Each indeterminate form can lead to different answers depending on the functions involved.

Example 1: The Form $\frac{0}{0}$

$$\lim_{x\to 0}\frac{\sin x}{x}=1,$$

while

$$\lim_{x\to 0}\frac{x^2}{x}=0,$$

and

$$\lim_{x\to 0}\frac{x}{x^2}=\infty.$$

All three limits produce

$$\frac{0}{0},$$

yet the limits are different.

Example 2: The Form $\frac{\infty}{\infty}$ Consider

$$\lim_{x\to \infty}\frac{x}{x}=1,$$

but

$$\lim_{x\to \infty}\frac{x^2}{x}=\infty,$$

and

$$\lim_{x\to \infty}\frac{x}{x^2}=0.$$

Again, each limit begins with the same indeterminate form

$$\frac{\infty}{\infty},$$

yet produces different values.

Example 3: The Form $0\cdot\infty$ As

$$x\to 0^+,$$

we have

$$x\to 0$$

while

$$\ln x\to -\infty.$$

Consequently,

$$x\ln x$$

has the indeterminate form

$$0(-\infty).$$

Later in this lecture we will show that

$$\lim_{x\to 0^+} x\ln x = 0.$$

3.5 Recognizing Indeterminate Forms

When evaluating a limit, always begin by substituting the limiting value into the expression.
There are three possible outcomes:

1. The substitution produces a finite number.

   The limit is immediately determined by continuity.

2. The substitution produces an undefined but non-indeterminate quantity, such as

   $$\frac{5}{0},$$

   which usually indicates that the limit is infinite or does not exist.

3. The substitution produces one of the seven indeterminate forms.

   Further work is required.

Recognizing an indeterminate form is the first step toward selecting the appropriate technique.

3.6 Techniques for Resolving Indeterminate Forms

Several methods can be used to evaluate indeterminate limits.
Depending on the problem, one may use

- algebraic simplification,
- factoring,
- rationalization,
- trigonometric identities,
- series expansions,
- squeeze theorem,
- or **L'Hôpital's Rule**.

Among these methods, L'Hôpital's Rule is one of the most powerful tools for limits involving the forms

$$\frac{0}{0} \quad \text{and} \quad \frac{\infty}{\infty}.$$

The next section develops this theorem and explains when it may be applied.

4. L'Hôpital's Rule

One of the most powerful techniques for evaluating limits involving indeterminate forms is **L'Hôpital's Rule**. This theorem allows us to replace a difficult limit with a simpler one by differentiating the numerator and denominator separately.
It is important to remember that L'Hôpital's Rule is not a general method for computing all limits. It applies only under specific conditions, and these conditions must always be verified before using the theorem.

4.1 Statement of L'Hôpital's Rule

Theorem 4.1 (L'Hôpital's Rule)**.** *Suppose that*

$$\lim_{x\to a} f(x) = 0, \qquad \lim_{x\to a} g(x) = 0,$$

or

$$\lim_{x\to a} |f(x)| = \infty, \qquad \lim_{x\to a} |g(x)| = \infty.$$

Assume that

- *f and g are differentiable on an open interval containing a (except possibly at a),*
- *$g'(x) \neq 0$ near a,*
- *the limit*

  $$\lim_{x\to a} \frac{f'(x)}{g'(x)}$$

  exists or is equal to $\pm\infty$.

Then

$$\boxed{\lim_{x\to a} \frac{f(x)}{g(x)} = \lim_{x\to a} \frac{f'(x)}{g'(x)}.}$$

4.2 When Can L'Hôpital's Rule Be Used?

L'Hôpital's Rule applies only to limits that initially produce one of the following indeterminate forms:

$$\boxed{\frac{0}{0}} \qquad \text{or} \qquad \boxed{\frac{\infty}{\infty}}.$$

If direct substitution produces any other expression, the rule cannot be applied immediately.
For example,

$$\frac{5}{0}, \qquad \frac{0}{3}, \qquad 7,$$

are *not* indeterminate forms, so L'Hôpital's Rule is not appropriate.
Other indeterminate forms such as

$$0\cdot\infty, \qquad \infty-\infty, \qquad 1^{\infty}, \qquad 0^0, \qquad \infty^0,$$

must first be rewritten as either

$$\frac{0}{0}$$

or

$$\frac{\infty}{\infty}.$$

4.3 Why the Rule Works

Although a complete proof of L'Hôpital's Rule requires advanced results from analysis (specifically Cauchy's Mean Value Theorem), the underlying idea is quite intuitive.

Near the point of interest, differentiable functions are well approximated by their tangent lines.
If both the numerator and denominator approach zero (or both become unbounded), the ratio of the functions is approximately equal to the ratio of their rates of change.
Thus,

$$\frac{f(x)}{g(x)} \approx \frac{f'(x)}{g'(x)},$$

which explains why differentiating the numerator and denominator often preserves the value of the limit.

4.4 Algorithm for Applying L'Hôpital's Rule

Whenever you encounter a limit involving a quotient, follow the steps below.

1. Evaluate the limit by direct substitution.
2. Determine whether the result is

   $$\frac{0}{0} \quad \text{or} \quad \frac{\infty}{\infty}.$$

   If not, stop and use another method.
3. Differentiate the numerator.
4. Differentiate the denominator.
5. Form the new quotient

   $$\frac{f'(x)}{g'(x)}.$$

6. Evaluate the new limit.
7. If another indeterminate form appears, L'Hôpital's Rule may be applied again.

4.5 Important Remarks

Students often make mistakes when applying L'Hôpital's Rule. Keep the following points in mind.

1. Differentiate the numerator and denominator separately.

   Do *not* differentiate the quotient using the Quotient Rule.

   For example,

   $$\boxed{\frac{d}{dx}\left(\frac{f(x)}{g(x)}\right) \neq \frac{f'(x)}{g'(x)}.}$$

   L'Hôpital's Rule is a theorem about limits, not derivatives.
2. Verify that an indeterminate form exists before differentiating.

   Never apply the rule simply because the limit looks difficult.
3. L'Hôpital's Rule may be applied more than once.

   Some limits require two or three successive differentiations before the indeterminate form disappears.
4. After each differentiation, evaluate the limit again.

   Do not continue differentiating unless another indeterminate form appears.

5. Algebraic simplification is often easier than using L'Hôpital's Rule.

   Always look for common factors, identities, or cancellations before differentiating.

4.6 Common Mistakes

The following are among the most common errors made by students.

Mistake 1. Applying L'Hôpital's Rule before checking the limit.

Mistake 2. Differentiating only the numerator.

Mistake 3. Differentiating the quotient using the Quotient Rule instead of differentiating the numerator and denominator separately.

Mistake 4. Applying the rule to limits that are not indeterminate.

Mistake 5. Forgetting that algebraic simplification may eliminate the indeterminate form without any differentiation.

4.7 Summary

L'Hôpital's Rule is one of the most effective techniques for evaluating limits involving indeterminate quotients.
Its use may be summarized by the following decision process:

Evaluate the limit by substitution
$\downarrow$
Is the result $\frac{0}{0}$ or $\frac{\infty}{\infty}$?
$\downarrow$
Differentiate numerator and denominator
$\downarrow$
Evaluate the new limit
$\downarrow$
Repeat if necessary

In the following sections, we apply this theorem to a variety of limits, illustrating when it succeeds and when algebraic manipulation is a better choice.

5. Applications of L'Hôpital's Rule

The best way to understand L'Hôpital's Rule is through carefully worked examples. In each example, we first verify that the hypotheses of the theorem are satisfied before differentiating the numerator and denominator.

5.1 Example 1: A $\frac{0}{0}$ Indeterminate Form

Evaluate

$$\lim_{x \to 0} \frac{\sin x}{x}.$$

This is one of the most fundamental limits in calculus. Although it can be proved geometrically or by the Squeeze Theorem, it also provides an excellent illustration of L'Hôpital's Rule.

Step 1. Evaluate by Direct Substitution Substituting $x = 0$ gives

$$\frac{\sin 0}{0} = \frac{0}{0}.$$

Since this is an indeterminate form, L'Hôpital's Rule applies.

Step 2. Differentiate Numerator and Denominator Differentiate each function separately. The derivative of the numerator is

$$\frac{d}{dx}(\sin x) = \cos x.$$

The derivative of the denominator is

$$\frac{d}{dx}(x) = 1.$$

Therefore,

$$\lim_{x \to 0} \frac{\sin x}{x} = \lim_{x \to 0} \frac{\cos x}{1}.$$

Step 3. Evaluate the New Limit Since cosine is continuous,

$$\lim_{x \to 0} \cos x = \cos 0 = 1.$$

Hence,

$$\boxed{\lim_{x \to 0} \frac{\sin x}{x} = 1.}$$

Discussion This example illustrates the simplest possible application of L'Hôpital's Rule. One differentiation completely removes the indeterminate form.

5.2 Example 2: Exponential and Logarithmic Functions

Evaluate

$$\lim_{x \to 0} \frac{e^x - 1}{x}.$$

Step 1. Direct Substitution Substituting $x = 0$,

$$\frac{e^0 - 1}{0} = \frac{0}{0}.$$

Again, the limit has the indeterminate form $\frac{0}{0}$.

Step 2. Apply L'Hôpital's Rule Differentiate the numerator:

$$\frac{d}{dx}(e^x - 1) = e^x.$$

Differentiate the denominator:

$$\frac{d}{dx}(x) = 1.$$

Thus,

$$\lim_{x\to 0} \frac{e^x - 1}{x} = \lim_{x\to 0} e^x.$$

Step 3. Evaluate Since the exponential function is continuous,

$$\lim_{x\to 0} e^x = e^0 = 1.$$

Therefore,

$$\boxed{\lim_{x\to 0} \frac{e^x - 1}{x} = 1.}$$

Interpretation Notice that this limit is exactly the derivative of the exponential function evaluated at $x = 0$:

$$\frac{d}{dx}(e^x)\Big|_{x=0} = 1.$$

5.3 Example 3: A Rational Function

Evaluate

$$\lim_{x\to 1} \frac{x^2 - 1}{x - 1}.$$

Method 1: Algebraic Simplification Factor the numerator:

$$x^2 - 1 = (x - 1)(x + 1).$$

For $x \neq 1$,

$$\frac{x^2 - 1}{x - 1} = x + 1.$$

Therefore,

$$\lim_{x\to 1}(x + 1) = 2.$$

Method 2: L'Hôpital's Rule Substituting $x = 1$,

$$\frac{1^2 - 1}{1 - 1} = \frac{0}{0}.$$

Differentiate numerator and denominator:

$$\frac{d}{dx}(x^2-1)=2x,$$

$$\frac{d}{dx}(x-1)=1.$$

Hence,

$$\lim_{x\to 1}\frac{x^2-1}{x-1}=\lim_{x\to 1}2x=2.$$

Thus,

$$\boxed{\lim_{x\to 1}\frac{x^2-1}{x-1}=2.}$$

Remark Although L'Hôpital's Rule works, algebraic simplification is much shorter. Whenever possible, simplify first before differentiating.

5.4 Example 4: Repeated Applications of L'Hôpital's Rule

Evaluate

$$\lim_{x\to 0}\frac{1-\cos x}{x^2}.$$

Step 1. Direct Substitution Substituting $x=0$,

$$\frac{1-\cos 0}{0}=\frac{0}{0}.$$

Apply L'Hôpital's Rule.

Step 2. First Differentiation Differentiate the numerator:

$$\frac{d}{dx}(1-\cos x)=\sin x.$$

Differentiate the denominator:

$$\frac{d}{dx}(x^2)=2x.$$

The limit becomes

$$\lim_{x\to 0}\frac{\sin x}{2x}.$$

Substituting again,

$$\frac{0}{0}.$$

The indeterminate form remains.

Step 3. Apply L'Hôpital's Rule Again Differentiate again.
Numerator:

$$\cos x.$$

Denominator:

$$2.$$

Hence,

$$\lim_{x \to 0} \frac{\cos x}{2} = \frac{1}{2}.$$

Therefore,

$$\boxed{\lim_{x \to 0} \frac{1 - \cos x}{x^2} = \frac{1}{2}.}$$

Discussion This example demonstrates that L'Hôpital's Rule may be applied repeatedly. After each differentiation, always check whether the new limit is still indeterminate before differentiating again.

5.5 Summary of the First Examples

The previous examples illustrate several important ideas.

1. Always begin by checking the limit through direct substitution.
2. Apply L'Hôpital's Rule only when the limit has the form

$$\frac{0}{0} \quad \text{or} \quad \frac{\infty}{\infty}.$$

3. Differentiate the numerator and denominator separately.
4. Re-evaluate the limit after every differentiation.
5. If another indeterminate form appears, apply the rule again.
6. If algebraic simplification removes the indeterminate form, there is no need to use L'Hôpital's Rule.

5.6 Example 1: A Logarithmic Limit

Evaluate

$$\lim_{x \to 1} \frac{\ln x}{x - 1}.$$

This limit appears frequently in calculus because it is closely related to the derivative of the natural logarithm function. Direct substitution gives

$$\frac{\ln(1)}{1 - 1} = \frac{0}{0},$$

which is an indeterminate form. Therefore, L'Hôpital's Rule may be applied.

Step 1. Differentiate the Numerator Since

$$\frac{d}{dx}(\ln x) = \frac{1}{x},$$

the numerator becomes

$$\frac{1}{x}.$$

Step 2. Differentiate the Denominator Since

$$\frac{d}{dx}(x-1) = 1,$$

the denominator becomes simply

$$1.$$

Therefore,

$$\lim_{x\to 1} \frac{\ln x}{x-1} = \lim_{x\to 1} \frac{1/x}{1} = \lim_{x\to 1} \frac{1}{x}.$$

Step 3. Evaluate the Limit Because

$$\frac{1}{x}$$

is continuous at $x = 1$,

$$\lim_{x\to 1} \frac{1}{x} = 1.$$

Hence,

$$\boxed{\lim_{x\to 1} \frac{\ln x}{x-1} = 1.}$$

Interpretation Notice that this limit is exactly the derivative of

$$\ln x$$

at

$$x = 1.$$

Indeed,

$$\frac{d}{dx}(\ln x) = \frac{1}{x},$$

so

$$\frac{d}{dx}(\ln x)\Big|_{x=1} = 1.$$

This provides another way to verify the result.

5.7 Example 2: An $\frac{\infty}{\infty}$ Indeterminate Form

Evaluate

$$\lim_{x\to\infty} \frac{\ln x}{x^{1/3}}.$$

As

$$x \to \infty,$$

both the numerator and denominator increase without bound.
Therefore,

$$\frac{\ln x}{x^{1/3}}$$

has the indeterminate form

$$\frac{\infty}{\infty},$$

so L'Hôpital's Rule applies.

Step 1. Differentiate the Numerator Since

$$\frac{d}{dx}(\ln x) = \frac{1}{x},$$

the numerator becomes

$$\frac{1}{x}.$$

Step 2. Differentiate the Denominator Using the power rule,

$$\frac{d}{dx}(x^{1/3}) = \frac{1}{3}x^{-2/3}.$$

Thus,

$$\lim_{x\to\infty} \frac{\ln x}{x^{1/3}} = \lim_{x\to\infty} \frac{\frac{1}{x}}{\frac{1}{3}x^{-2/3}}.$$

Step 3. Simplify Multiply numerator and denominator:

$$= \lim_{x\to\infty} 3x^{-1}x^{2/3}.$$

Combining exponents,

$$= \lim_{x\to\infty} \frac{3}{x^{1/3}}.$$

Step 4. Evaluate Since

$$x^{1/3} \to \infty,$$

we obtain

$$\boxed{\lim_{x\to\infty} \frac{\ln x}{x^{1/3}} = 0.}$$

Discussion This example illustrates an important principle.
Although

$$\ln x$$

increases without bound, every positive power of

$$x$$

grows much faster.
Consequently,

$$\boxed{\ln x = o(x^{\alpha}) \qquad (\alpha > 0).}$$

This growth comparison appears frequently throughout calculus and advanced analysis.

5.8 Example 3: Repeated Application of L'Hôpital's Rule

Evaluate

$$\lim_{x\to 0} \frac{\tan x - x}{x^3}.$$

This limit is more challenging because the indeterminate form persists after the first differentiation. We will apply L'Hôpital's Rule repeatedly until the limit can be evaluated.

Step 1. Direct Substitution Substituting $x = 0$,

$$\frac{\tan 0 - 0}{0^3} = \frac{0}{0}.$$

Since the limit has the indeterminate form

$$\frac{0}{0},$$

L'Hôpital's Rule applies.

Step 2. First Application Differentiate the numerator:

$$\frac{d}{dx}(\tan x - x) = \sec^2 x - 1.$$

Differentiate the denominator:

$$\frac{d}{dx}(x^3) = 3x^2.$$

Thus,

$$\lim_{x\to 0} \frac{\tan x - x}{x^3} = \lim_{x\to 0} \frac{\sec^2 x - 1}{3x^2}.$$

Substituting $x = 0$,

$$\frac{1-1}{0} = \frac{0}{0}.$$

The limit is still indeterminate.

Step 3. Second Application Differentiate again.
Using

$$\frac{d}{dx}(\sec^2 x) = 2\sec^2 x \tan x,$$

we obtain

$$\frac{d}{dx}(\sec^2 x - 1) = 2\sec^2 x \tan x.$$

The denominator becomes

$$6x.$$

Hence,

$$\lim_{x\to 0} \frac{2\sec^2 x \tan x}{6x}.$$

Again,

$$\frac{0}{0}.$$

A third application is required.

Step 4. Third Application Differentiate the numerator carefully using the Product Rule.
Let

$$u = \sec^2 x, \qquad v = \tan x.$$

Then

$$u' = 2\sec^2 x \tan x,$$

and

$$v' = \sec^2 x.$$

Therefore,

$$\begin{aligned}\frac{d}{dx}\left(2\sec^2 x \tan x\right) &= 2\left[(2\sec^2 x \tan x)\tan x + \sec^4 x\right] \\ &= 4\sec^2 x \tan^2 x + 2\sec^4 x.\end{aligned}$$

The denominator becomes

$$6.$$

Thus,

$$\lim_{x\to 0} \frac{4\sec^2 x \tan^2 x + 2\sec^4 x}{6}.$$

Step 5. Evaluate Since

$$\tan 0 = 0, \qquad \sec 0 = 1,$$

we obtain

$$\frac{4(1)(0) + 2(1)}{6} = \frac{2}{6} = \frac{1}{3}.$$

Therefore,

$$\boxed{\lim_{x \to 0} \frac{\tan x - x}{x^3} = \frac{1}{3}.}$$

Remark This example illustrates that L'Hôpital's Rule may need to be applied several times before the indeterminate form disappears. After every differentiation, always substitute again to determine whether another application is necessary.

5.9 Example 4: A Trigonometric Limit Near π

Evaluate

$$\lim_{x \to \pi^-} \frac{\sin x}{1 + \cos x}.$$

Step 1. Direct Substitution Since

$$\sin \pi = 0, \qquad \cos \pi = -1,$$

direct substitution gives

$$\frac{0}{0}.$$

Therefore,
L'Hôpital's Rule applies.

Step 2. Differentiate Differentiate the numerator:

$$\frac{d}{dx}(\sin x) = \cos x.$$

Differentiate the denominator:

$$\frac{d}{dx}(1 + \cos x) = -\sin x.$$

Hence,

$$\lim_{x \to \pi^-} \frac{\cos x}{-\sin x}.$$

Step 3. Evaluate As

$$x \to \pi^-,$$

we have

$$\cos \pi = -1,$$

while

$$\sin x \to 0^+.$$

Therefore,

$$-\sin x \to 0^-.$$

Consequently,

$$\frac{-1}{0^-} = +\infty.$$

Hence,

$$\boxed{\lim_{x \to \pi^-} \frac{\sin x}{1 + \cos x} = +\infty.}$$

Geometric Interpretation Near

$$x = \pi,$$

the denominator

$$1 + \cos x$$

approaches zero much faster than the numerator.
As a result, the quotient becomes arbitrarily large, and the limit diverges to positive infinity.

5.10 Example 5: Exponential Growth Versus Polynomial Growth

Evaluate

$$\lim_{x \to \infty} \frac{e^x}{x^2}.$$

Direct substitution gives

$$\frac{\infty}{\infty},$$

so L'Hôpital's Rule applies.
Differentiate once:

$$\lim_{x \to \infty} \frac{e^x}{2x}.$$

The limit is still

$$\frac{\infty}{\infty},$$

so differentiate again.

$$\lim_{x \to \infty} \frac{e^x}{2}.$$

Since

$$e^x \to \infty,$$

we conclude

$$\boxed{\lim_{x\to\infty} \frac{e^x}{x^2} = \infty.}$$

Important Observation This example illustrates one of the most important growth comparisons in calculus.
As

$$x \to \infty,$$

the exponential function grows faster than every polynomial.
Symbolically,

$$\boxed{x^n = o(e^x) \qquad (n > 0).}$$

This fact is used repeatedly throughout calculus, differential equations, and applied mathematics.

6. Other Indeterminate Forms

L'Hôpital's Rule applies directly only to the indeterminate forms

$$\boxed{\frac{0}{0}} \quad \text{and} \quad \boxed{\frac{\infty}{\infty}}.$$

Fortunately, every other indeterminate form encountered in calculus can be rewritten as one of these two forms. Once rewritten, L'Hôpital's Rule may be applied in exactly the same manner.
The most common additional indeterminate forms are

$$0 \cdot \infty, \qquad \infty - \infty, \qquad 1^\infty, \qquad 0^0, \qquad \infty^0.$$

Each requires a different algebraic transformation before differentiation.

6.1 The Indeterminate Form $0 \cdot \infty$

Suppose a limit has the form

$$0 \cdot \infty.$$

Since L'Hôpital's Rule cannot be applied directly to a product, the product must first be rewritten as a quotient.
Two useful identities are

$$fg = \frac{f}{1/g},$$

or

$$fg = \frac{g}{1/f}.$$

Choose whichever form produces either

$$\frac{0}{0}$$

or

$$\frac{\infty}{\infty}.$$

Example Evaluate

$$\lim_{x\to 0^+} x\ln x.$$

As

$$x \to 0^+,$$

we have

$$x \to 0, \qquad \ln x \to -\infty,$$

so the limit has the indeterminate form

$$0(-\infty).$$

Rewrite the product as

$$x\ln x = \frac{\ln x}{1/x}.$$

Now the limit becomes

$$\lim_{x\to 0^+} \frac{\ln x}{1/x},$$

which has the form

$$\frac{-\infty}{\infty}.$$

Apply L'Hôpital's Rule.
Differentiate the numerator:

$$\frac{d}{dx}(\ln x) = \frac{1}{x}.$$

Differentiate the denominator:

$$\frac{d}{dx}\left(\frac{1}{x}\right) = -\frac{1}{x^2}.$$

Hence,

$$\begin{aligned}\lim_{x\to 0^+} \frac{\ln x}{1/x} &= \lim_{x\to 0^+} \frac{1/x}{-1/x^2}\\ &= \lim_{x\to 0^+} (-x)\\ &= 0.\end{aligned}$$

Therefore,

$$\boxed{\lim_{x \to 0^+} x \ln x = 0.}$$

6.2 The Indeterminate Form $\infty - \infty$

Expressions involving subtraction cannot be differentiated directly using L'Hôpital's Rule.
Instead, combine the terms into a single fraction by finding a common denominator or by rationalizing.

Example Evaluate

$$\lim_{x \to \infty} \left(\sqrt{x^2 + x} - x \right).$$

Direct substitution gives

$$\infty - \infty.$$

Multiply by the conjugate:

$$\begin{aligned} \sqrt{x^2 + x} - x &= \frac{(\sqrt{x^2 + x} - x)(\sqrt{x^2 + x} + x)}{\sqrt{x^2 + x} + x} \\ &= \frac{x}{\sqrt{x^2 + x} + x}. \end{aligned}$$

The limit becomes

$$\lim_{x \to \infty} \frac{x}{\sqrt{x^2 + x} + x},$$

which is now of the form

$$\frac{\infty}{\infty}.$$

L'Hôpital's Rule could now be applied if desired.
(An even shorter approach is to divide numerator and denominator by x.)

6.3 Exponential Indeterminate Forms

The remaining indeterminate forms involve exponents:

$$1^\infty, \qquad 0^0, \qquad \infty^0.$$

These cannot be differentiated directly.
Instead, logarithms are used to convert the exponent into a product.

General Strategy Suppose

$$y = f(x)^{g(x)}.$$

Take the natural logarithm of both sides:

$$\ln y = g(x) \ln(f(x)).$$

Now evaluate

$$\lim \ln y$$

using algebra and L'Hôpital's Rule if necessary.
Finally,

$$y = e^{\ln y},$$

so exponentiating gives the desired limit.

6.4 Example: The Form 1^∞

Evaluate

$$\lim_{x\to\infty} \left(1 + \frac{1}{x}\right)^x.$$

Direct substitution gives

$$1^\infty.$$

Let

$$y = \left(1 + \frac{1}{x}\right)^x.$$

Taking logarithms,

$$\ln y = x \ln \left(1 + \frac{1}{x}\right).$$

Rewrite as

$$\ln y = \frac{\ln(1 + 1/x)}{1/x}.$$

As

$$x \to \infty,$$

both numerator and denominator approach zero, producing the form

$$\frac{0}{0}.$$

Apply L'Hôpital's Rule.
The derivative of the numerator is

$$\frac{-1/x^2}{1 + 1/x},$$

while the derivative of the denominator is

$$-\frac{1}{x^2}.$$

Therefore,

$$\lim_{x\to\infty} \ln y = 1.$$

Exponentiating,

$$\boxed{\lim_{x\to\infty}\left(1+\frac{1}{x}\right)^x = e.}$$

6.5 Summary of the Indeterminate Forms

The following table summarizes how each indeterminate form should be handled.

Form	**Strategy**
$\frac{0}{0}$	Apply L'Hôpital's Rule
$\frac{\infty}{\infty}$	Apply L'Hôpital's Rule
$0\cdot\infty$	Rewrite as a quotient
$\infty-\infty$	Combine into one fraction
1^∞	Take logarithms
0^0	Take logarithms
∞^0	Take logarithms

This completes the discussion of L'Hôpital's Rule and the various indeterminate forms. We now turn our attention to another important topic in integration: **improper integrals**, which arise whenever an interval is unbounded or the integrand becomes unbounded on the interval.