Calculus II

Lecture 4

§7.8 Improper Integrals, §8.1 Arc Length & §8.3 Area of a Surface of Revolution

Contents

1. Improper Integrals

In previous sections, every definite integral that we evaluated satisfied two important conditions:

1. The interval of integration was finite.
2. The integrand was continuous on the entire interval.

For example,

$$\int_0^2 (x^2 + 1)\,dx$$

is a proper definite integral because

- the interval

  $$[0, 2]$$

  is finite, and

- the function

  $$x^2 + 1$$

  is continuous everywhere.

However, many important applications in mathematics, physics, probability, and engineering involve integrals for which one or both of these conditions fail.
Examples include

$$\int_1^\infty \frac{1}{x^2}\,dx,$$

$$\int_0^1 \frac{1}{\sqrt{x}}\,dx,$$

and

$$\int_{-\infty}^\infty e^{-x^2}\,dx.$$

These integrals are called **improper integrals** because the interval of integration is infinite or because the integrand becomes unbounded on the interval.

1.1 Why Do We Need Improper Integrals?

Many real-world problems require us to integrate over regions that extend without bound.
Examples include

- total probability of continuous distributions,
- electric and gravitational fields,
- Fourier transforms,
- Laplace transforms,
- infinite physical domains.

Similarly, some functions become infinite near certain points but still enclose a finite amount of area.
Consequently, the ordinary definition of the definite integral is not sufficient.
Instead, improper integrals are defined using limits.

1.2 Definition of an Improper Integral

An improper integral is a definite integral in which at least one of the following occurs.

1. One or both limits of integration are infinite.
2. The integrand becomes unbounded at one or more points of the interval.

In either case, the improper integral is interpreted as the limit of a proper definite integral.

1.3 Two Types of Improper Integrals

Improper integrals are divided into two categories.

Type I The interval of integration is infinite.
Examples include

$$\int_1^\infty \frac{1}{x^2}\,dx,$$

$$\int_{-\infty}^3 e^x\,dx,$$

and

$$\int_{-\infty}^\infty \frac{1}{1+x^2}\,dx.$$

Type II The interval of integration is finite, but the integrand becomes infinite somewhere on the interval.
Examples include

$$\int_0^1 \frac{1}{\sqrt{x}}\,dx,$$

$$\int_0^1 \frac{1}{x}\,dx,$$

and

$$\int_{-2}^2 \frac{1}{x^2}\,dx.$$

1.4 Convergence and Divergence

Unlike ordinary definite integrals, improper integrals do not always have a finite value.

Definition 1.1. An improper integral is said to **converge** if the defining limit exists and is finite. It is said to **diverge** if the limit does not exist or is infinite.

For example,

$$\int_1^\infty \frac{1}{x^2}\,dx$$

converges,
whereas

$$\int_1^\infty \frac{1}{x}\,dx$$

diverges.
Determining whether an improper integral converges is one of the principal goals of this section.

1.5 General Strategy

Every improper integral should be approached using the same general procedure.

1. Identify why the integral is improper.
2. Replace the infinite limit or discontinuity by a variable.
3. Rewrite the integral as a limit.
4. Evaluate the resulting definite integral.
5. Compute the limit.
6. Decide whether the integral converges or diverges.

Notice that every improper integral is ultimately transformed into a limit problem.
This explains why our discussion of L'Hôpital's Rule naturally precedes the study of improper integrals.

1.6 Notation

Whenever an integral contains

$$\infty, \qquad -\infty,$$

or an infinite discontinuity,
remember that these symbols are not actual numbers.
For example,

$$\int_1^\infty f(x)\,dx$$

does *not* mean that we substitute

$$\infty$$

into the antiderivative.
Instead, the symbol

$$\infty$$

indicates that a limit must be evaluated.
This distinction is extremely important and is the basis for the rigorous definition of improper integrals developed in the next sections.

2. Type I Improper Integrals

A **Type I improper integral** is an integral whose interval of integration extends to positive infinity, negative infinity, or both.
Since infinity is not a real number, the Fundamental Theorem of Calculus cannot be applied directly. Instead, the improper integral is defined as the limit of a family of proper definite integrals.

2.1 Infinite Upper Limit

Consider the integral

$$\int_a^\infty f(x)\,dx.$$

The symbol

$$\infty$$

does not represent a number. Therefore, we replace it with a variable upper limit t, evaluate the definite integral, and then let t approach positive infinity.

Definition 2.1. The improper integral

$$\int_a^\infty f(x)\,dx$$

is defined by

$$\boxed{\int_a^\infty f(x)\,dx = \lim_{t\to\infty} \int_a^t f(x)\,dx,}$$

provided that the limit exists.
If the limit exists and is finite, the integral is said to **converge**.
Otherwise, it is said to **diverge**.

2.2 Infinite Lower Limit

Similarly,

$$\int_{-\infty}^b f(x)\,dx$$

is defined by replacing the lower limit with a variable.

Definition 2.2.

$$\boxed{\int_{-\infty}^b f(x)\,dx = \lim_{t\to-\infty} \int_t^b f(x)\,dx.}$$

Again, the improper integral converges if the limit exists and diverges otherwise.

2.3 Infinite Limits on Both Ends

Sometimes both limits are infinite.
For example,

$$\int_{-\infty}^\infty f(x)\,dx.$$

This integral must never be evaluated by replacing both limits simultaneously.
Instead, choose any convenient number c and split the integral into two parts.

$$\boxed{\int_{-\infty}^\infty f(x)\,dx = \int_{-\infty}^c f(x)\,dx + \int_c^\infty f(x)\,dx.}$$

Each integral is evaluated separately.
The original improper integral converges only if both integrals converge.

2.4 Example 1: A Convergent Improper Integral

Evaluate

$$\int_1^\infty \frac{1}{x^2}\,dx.$$

Step 1. Rewrite as a Limit Using the definition,

$$\int_1^\infty \frac{1}{x^2}\,dx = \lim_{t\to\infty} \int_1^t \frac{1}{x^2}\,dx.$$

Step 2. Evaluate the Definite Integral Rewrite the integrand.

$$\frac{1}{x^2} = x^{-2}.$$

Applying the Power Rule,

$$\int x^{-2}\,dx = -\frac{1}{x}.$$

Hence,

$$\begin{aligned}\int_1^t \frac{1}{x^2}\,dx &= \left[-\frac{1}{x}\right]_1^t \\ &= -\frac{1}{t} + 1.\end{aligned}$$

Step 3. Compute the Limit Taking the limit,

$$\lim_{t\to\infty}\left(1 - \frac{1}{t}\right) = 1.$$

Therefore,

$$\boxed{\int_1^\infty \frac{1}{x^2}\,dx = 1.}$$

Conclusion Since the limit exists and is finite, the improper integral **converges**.

2.5 Example 2: A Divergent Improper Integral

Evaluate

$$\int_1^\infty \frac{1}{x}\,dx.$$

Step 1. Rewrite as a Limit By definition,

$$\int_1^\infty \frac{1}{x}\,dx = \lim_{t\to\infty}\int_1^t \frac{1}{x}\,dx.$$

Step 2. Integrate Since

$$\int \frac{1}{x}\,dx = \ln|x|,$$

we obtain

$$\begin{aligned}\int_1^t \frac{1}{x}\,dx &= [\ln x]_1^t \\ &= \ln t.\end{aligned}$$

Step 3. Evaluate the Limit As

$$t \to \infty,$$

$$\ln t \to \infty.$$

Therefore,

$$\boxed{\int_1^\infty \frac{1}{x}\,dx = \infty.}$$

The improper integral diverges.

Discussion Although

$$\frac{1}{x} \to 0$$

as

$$x \to \infty,$$

the area under the curve is still infinite.
This is an important observation.
A function approaching zero does *not* guarantee that the corresponding improper integral converges.

2.6 Example 3: Exponential Decay

Evaluate

$$\int_0^\infty e^{-x}\,dx.$$

Step 1. Rewrite Using the Definition

$$\int_0^\infty e^{-x}\,dx = \lim_{t\to\infty}\int_0^t e^{-x}\,dx.$$

Step 2. Integrate Since

$$\int e^{-x}\,dx = -e^{-x},$$

we have

$$\begin{aligned}\int_0^t e^{-x}\,dx &= \left[-e^{-x}\right]_0^t \\ &= 1 - e^{-t}.\end{aligned}$$

Step 3. Evaluate the Limit Because

$$e^{-t} \to 0 \qquad (t \to \infty),$$

we conclude

$$\boxed{\int_0^\infty e^{-x}\,dx = 1.}$$

Thus, the improper integral converges.

2.7 Important Remarks

These examples illustrate several key ideas.

1. Every Type I improper integral must first be rewritten as a limit.
2. Never substitute

$$\infty$$

   directly into an antiderivative.
3. The convergence of an improper integral depends on the value of the limit, not merely on the behavior of the integrand.
4. A function approaching zero is not sufficient for convergence.
5. Exponentially decaying functions generally produce convergent improper integrals.

3. Type II Improper Integrals

Unlike Type I improper integrals, which involve infinite intervals, a **Type II improper integral** is defined on a finite interval but has an integrand that becomes unbounded (infinite) at one or more points of the interval.
Typical situations include functions having

$$\frac{1}{x}, \qquad \frac{1}{\sqrt{x}}, \qquad \frac{1}{(x-a)^2},$$

or other vertical asymptotes.
Since the function is not continuous on the interval, the Fundamental Theorem of Calculus cannot be applied directly.
Instead, the improper integral is defined by an appropriate limit.

3.1 Improper Integrals with a Discontinuity at an Endpoint

Suppose that

$$f(x)$$

becomes unbounded at the lower endpoint

$$x = a.$$

Then

$$\int_a^b f(x)\,dx$$

is defined by replacing the endpoint with a variable.

Definition 3.1. If

$$f(x) \to \infty \quad \text{as} \quad x \to a^+,$$

then

$$\boxed{\int_a^b f(x)\,dx = \lim_{t \to a^+} \int_t^b f(x)\,dx.}$$

If the limit exists and is finite, the integral converges.
Otherwise, it diverges.

3.2 Discontinuity at the Upper Endpoint

Similarly,
if

$$f(x)$$

becomes infinite at the upper endpoint,
then

$$\boxed{\int_a^b f(x)\,dx = \lim_{t\to b^-}\int_a^t f(x)\,dx.}$$

Again, convergence depends entirely upon the existence of the limit.

3.3 Discontinuity Inside the Interval

Sometimes the integrand becomes infinite at an interior point

$$c, \qquad a < c < b.$$

In this situation the integral must be separated into two independent improper integrals.

$$\boxed{\int_a^b f(x)\,dx = \int_a^c f(x)\,dx + \int_c^b f(x)\,dx.}$$

Each integral is evaluated separately using limits.
The original improper integral converges only if both improper integrals converge.

3.4 Example 1: A Convergent Type II Improper Integral

Evaluate

$$\int_0^1 \frac{1}{\sqrt{x}}\,dx.$$

Notice that

$$\frac{1}{\sqrt{x}} = x^{-1/2},$$

which becomes infinite as

$$x \to 0^+.$$

Therefore, the integral is improper.

Step 1. Rewrite Using the Definition Replace the lower limit by

$$t.$$

$$\int_0^1 \frac{1}{\sqrt{x}}\,dx = \lim_{t\to 0^+}\int_t^1 x^{-1/2}\,dx.$$

Step 2. Evaluate the Integral Using the Power Rule,

$$\int x^{-1/2}\,dx = 2x^{1/2}.$$

Hence,

$$\begin{aligned}\int_t^1 x^{-1/2}\,dx &= \left[2\sqrt{x}\right]_t^1\\ &= 2 - 2\sqrt{t}.\end{aligned}$$

Step 3. Compute the Limit As

$$t \to 0^+,$$

$$\sqrt{t} \to 0.$$

Therefore,

$$\boxed{\int_0^1 \frac{1}{\sqrt{x}}\,dx = 2.}$$

Thus the improper integral converges.

3.5 Example 2: A Divergent Type II Improper Integral

Evaluate

$$\int_0^1 \frac{1}{x}\,dx.$$

Since

$$\frac{1}{x}$$

becomes infinite at

$$x = 0,$$

the integral is improper.

Step 1. Rewrite Using the Definition

$$\int_0^1 \frac{1}{x}\,dx = \lim_{t\to 0^+} \int_t^1 \frac{1}{x}\,dx.$$

Step 2. Integrate Using

$$\int \frac{1}{x}\,dx = \ln|x|,$$

we obtain

$$\int_t^1 \frac{1}{x}\,dx = \ln 1 - \ln t$$
$$= -\ln t.$$

Step 3. Evaluate the Limit Since

$$\ln t \to -\infty \quad (t \to 0^+),$$

we have

$$-\ln t \to \infty.$$

Hence,

$$\boxed{\int_0^1 \frac{1}{x}\,dx = \infty.}$$

The improper integral diverges.

3.6 Example 3: A Discontinuity Inside the Interval

Evaluate

$$\int_{-1}^1 \frac{1}{x^2}\,dx.$$

The function

$$\frac{1}{x^2}$$

has a vertical asymptote at

$$x = 0,$$

which lies inside the interval.

Step 1. Split the Integral Write

$$\int_{-1}^1 \frac{1}{x^2}\,dx = \int_{-1}^0 \frac{1}{x^2}\,dx + \int_0^1 \frac{1}{x^2}\,dx.$$

Each integral must be treated separately.

Step 2. Evaluate the Left Integral

$$\int_{-1}^0 \frac{1}{x^2}\,dx = \lim_{t\to 0^-} \int_{-1}^t x^{-2}\,dx$$
$$= \lim_{t\to 0^-} \left[-\frac{1}{x}\right]_{-1}^t$$
$$= \lim_{t\to 0^-} \left(-\frac{1}{t} - 1\right).$$

Since

$$-\frac{1}{t} \to \infty \quad (t \to 0^-),$$

the left integral diverges.

Step 3. Evaluate the Right Integral Similarly,

$$\begin{aligned} \int_0^1 \frac{1}{x^2}\,dx &= \lim_{t\to 0^+} \left[-\frac{1}{x}\right]_t^1 \\ &= \lim_{t\to 0^+} \left(\frac{1}{t} - 1\right). \end{aligned}$$

Again,

$$\frac{1}{t} \to \infty.$$

Therefore the right integral also diverges.
Consequently,

$$\boxed{\int_{-1}^1 \frac{1}{x^2}\,dx \text{ diverges.}}$$

3.7 Summary of Type II Improper Integrals

The examples above illustrate the general procedure for evaluating Type II improper integrals.

1. Locate every point where the integrand becomes unbounded.
2. Rewrite the improper integral as one or more limits.
3. Evaluate the corresponding proper definite integrals.
4. Compute the limits.
5. Determine whether the integral converges or diverges.

Notice that a finite interval does not guarantee a finite area.
If the function grows too rapidly near a singularity, the enclosed area may still be infinite.

4. The Comparison Theorem

In many cases, an improper integral cannot be evaluated exactly. Fortunately, it is often unnecessary to compute the integral explicitly in order to determine whether it converges.
Instead, we compare the given function with another function whose improper integral is already known.
This idea leads to one of the most important convergence tests for improper integrals.

4.1 Motivation

Suppose two nonnegative functions satisfy

$$0 \le f(x) \le g(x)$$

for every x in an interval.
If the larger function encloses only a finite amount of area, then the smaller function must also enclose a finite amount of area.
Similarly, if the smaller function already encloses an infinite amount of area, then the larger function must also have infinite area.

This simple geometric observation forms the basis of the Comparison Theorem.

4.2 Comparison Theorem

Theorem 4.1 (Comparison Theorem)**.** *Suppose*

$$0 \le f(x) \le g(x)$$

for all sufficiently large values of x.
Then

1. *If*

   $$\int_a^\infty g(x)\,dx$$

   converges,

   then

   $$\int_a^\infty f(x)\,dx$$

   also converges.

2. *If*

   $$\int_a^\infty f(x)\,dx$$

   diverges,

   then

   $$\int_a^\infty g(x)\,dx$$

   also diverges.

4.3 Choosing a Comparison Function

Choosing an appropriate comparison function is the most important part of using the Comparison Theorem.
Common comparison functions include

$$\frac{1}{x^p},$$

$$e^{-x},$$

$$\frac{1}{x},$$

and

$$\frac{1}{1+x^2}.$$

Generally,

$$\frac{1}{x^p}$$

is the most useful because its convergence behavior is completely understood.

4.4 The p-Integral Test

The improper integral

$$\int_1^\infty \frac{1}{x^p}\,dx$$

plays a central role in comparison tests.
Evaluating the integral gives the following result.

Theorem 4.2 (p-Integral Test)**.** *For*

$$\int_1^\infty \frac{1}{x^p}\,dx,$$

$$\boxed{\begin{aligned} &p > 1 \implies \textit{Converges}, \\ &p \le 1 \implies \textit{Diverges.} \end{aligned}}$$

This theorem provides the standard comparison for most improper integrals.

4.5 Example 1

Determine whether

$$\int_1^\infty \frac{1}{x^2+1}\,dx$$

converges.

Solution Observe that

$$x^2 + 1 > x^2,$$

which implies

$$\frac{1}{x^2+1} < \frac{1}{x^2}.$$

Since

$$\int_1^\infty \frac{1}{x^2}\,dx$$

converges,
the Comparison Theorem immediately implies

$$\boxed{\int_1^\infty \frac{1}{x^2+1}\,dx \text{ converges.}}$$

4.6 Example 2

Determine whether

$$\int_1^\infty \frac{1}{\sqrt{x}}\,dx$$

converges.

Solution Notice that

$$\frac{1}{\sqrt{x}} = \frac{1}{x^{1/2}}.$$

Here,

$$p = \frac{1}{2} < 1.$$

Therefore,

$$\boxed{\int_1^\infty \frac{1}{\sqrt{x}}\,dx \text{ diverges.}}$$

4.7 Example 3

Determine whether

$$\int_1^\infty \frac{x}{x^3+1}\,dx$$

converges.

Solution For large values of x,

$$x^3 + 1 > x^3.$$

Hence,

$$\frac{x}{x^3+1} < \frac{x}{x^3} = \frac{1}{x^2}.$$

Since

$$\int_1^\infty \frac{1}{x^2}\,dx$$

converges,
the Comparison Theorem gives

$$\boxed{\int_1^\infty \frac{x}{x^3+1}\,dx \text{ converges.}}$$

4.8 Common Mistakes

Students frequently make the following mistakes when using comparison.

1. Comparing with a function whose convergence is unknown.
2. Choosing the inequality in the wrong direction.
3. Forgetting that the functions must remain nonnegative.
4. Attempting to compare functions that change sign.
5. Forgetting that convergence of the larger function tells us about the smaller function, while divergence of the smaller function tells us about the larger function.

Lecture Summary

In this lecture we studied two important topics.
First, we introduced L'Hôpital's Rule, one of the most powerful techniques for evaluating limits involving indeterminate forms.
The rule applies only to

$$\frac{0}{0} \quad \text{and} \quad \frac{\infty}{\infty},$$

while the remaining indeterminate forms must first be rewritten into one of these two forms.
The second half of the lecture introduced improper integrals.
Unlike ordinary definite integrals, improper integrals are defined using limits because either the interval is unbounded or the integrand becomes unbounded.
Finally, we developed the Comparison Theorem, which provides a practical tool for determining convergence without explicitly evaluating an improper integral.
The key ideas from this lecture are summarized below.

L'Hôpital's Rule

$\frac{0}{0}, \frac{\infty}{\infty} \Longrightarrow$ Differentiate numerator and denominator

Improper Integrals

Rewrite as limits
Evaluate
Determine convergence

Comparison Theorem

$0 \leq f(x) \leq g(x)$
Compare with a known integral

These ideas form the foundation for many advanced topics in calculus, including infinite series, probability theory, differential equations, and mathematical analysis.

5. Arc Length

In previous chapters we learned how to determine many geometric quantities associated with curves, including areas enclosed by curves, volumes of solids, and average values of functions. Another important geometric quantity is the **length of a curve**.
Suppose a particle moves along a smooth curve from one point to another. Naturally, we would like to know the total distance traveled. Unlike the distance between two points measured by a straight line, the distance along a curved path is generally much more difficult to compute.
This distance is called the **arc length** of the curve.
Finding the exact length of a curve has numerous applications in science, engineering, computer graphics, robotics, architecture, and physics. Examples include

- determining the length of roads or railway tracks,
- measuring the distance traveled by a moving particle,
- designing bridges and arches,
- manufacturing curved pipes and cables,

- computing the length of coastlines or river paths.

Fortunately, calculus provides a systematic method for computing the exact length of many smooth curves.
Throughout this section we assume that the curve is sufficiently smooth, meaning that its derivative exists and is continuous on the interval of interest. This assumption guarantees that the arc length integral exists.

5.1 The Basic Idea

The key idea behind the arc length formula is surprisingly simple.
Instead of attempting to measure the entire curved path at once, we divide the curve into many very small pieces.
Each small piece is so short that it is nearly a straight line.
The length of each tiny segment can therefore be approximated using the Pythagorean Theorem.
Adding together the lengths of all these small line segments produces an approximation of the total arc length.
As the segments become smaller and smaller, this approximation becomes better and better.
Taking the limit as the number of segments approaches infinity leads to the exact arc length formula.
This idea is illustrated schematically below.

$$\text{Curve} \quad \longrightarrow \quad \text{Many small line segments} \quad \longrightarrow \quad \text{Take a limit}$$

Thus, the problem of finding the length of a curved path is reduced to the problem of evaluating a definite integral.

5.2 Review: Distance Formula

Recall that the distance between two points

$$(x_1, y_1) \quad \text{and} \quad (x_2, y_2)$$

is given by

$$d = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}.$$

If we denote

$$\Delta x = x_2 - x_1, \qquad \Delta y = y_2 - y_1,$$

then the distance formula becomes

$$\boxed{\Delta s = \sqrt{(\Delta x)^2 + (\Delta y)^2}.}$$

This simple geometric formula is the starting point for deriving the arc length formula.
In the next section we will allow

$$\Delta x$$

and

$$\Delta y$$

to become infinitesimally small, leading to the differential arc length

$$ds = \sqrt{dx^2 + dy^2},$$

which forms the basis of all arc length calculations.

6. Derivation of the Arc Length Formula

The arc length formula is obtained by approximating a smooth curve with many small line segments. As the number of segments increases, the approximation becomes more accurate. Taking the limit leads to the exact formula.

6.1 Approximating the Curve by Line Segments

Consider a smooth curve

$$y = f(x), \qquad a \leq x \leq b,$$

where the derivative

$$f'(x)$$

is continuous on the interval.
Divide the interval

$$[a, b]$$

into many small subintervals

$$a = x_0 < x_1 < x_2 < \cdots < x_n = b.$$

Each pair of consecutive points on the curve determines a short line segment.
For the i^{th} segment,

$$\Delta x = x_i - x_{i-1},$$

and

$$\Delta y = f(x_i) - f(x_{i-1}).$$

Since the segment is very short, its length is approximately equal to the distance between the two endpoints.
By the Pythagorean Theorem,

$$\Delta s \approx \sqrt{(\Delta x)^2 + (\Delta y)^2}.$$

Thus, each small piece of the curve behaves almost like the hypotenuse of a right triangle.

6.2 Using the Pythagorean Theorem

Factor out

$$(\Delta x)^2$$

from inside the square root.

$$\begin{aligned}\Delta s &\approx \sqrt{(\Delta x)^2 + (\Delta y)^2} \\ &= \sqrt{(\Delta x)^2\left(1 + \left(\frac{\Delta y}{\Delta x}\right)^2\right)} \\ &= \sqrt{1 + \left(\frac{\Delta y}{\Delta x}\right)^2}\,\Delta x.\end{aligned}$$

Notice that

$$\frac{\Delta y}{\Delta x}$$

is the average rate of change of the function over the interval.
As the interval becomes smaller,

$$\frac{\Delta y}{\Delta x}$$

approaches the derivative.

6.3 Passing to the Limit

When the partition becomes infinitely fine,

$$\Delta x \to dx,$$

and

$$\frac{\Delta y}{\Delta x} \longrightarrow \frac{dy}{dx}.$$

Therefore,

$$ds = \sqrt{1 + \left(\frac{dy}{dx}\right)^2}\,dx.$$

The total arc length is obtained by summing all of the differential arc lengths.
In the limit, this sum becomes the definite integral

$$\boxed{L = \int_a^b \sqrt{1 + \left(\frac{dy}{dx}\right)^2}\,dx.}$$

This is the standard arc length formula for curves expressed in the form

$$y = f(x).$$

6.4 Interpretation of the Formula

Each quantity appearing in the formula has a clear geometric meaning.

- The quantity

$$dx$$

represents an infinitesimally small horizontal displacement.

- The quantity

$$dy$$

represents the corresponding infinitesimally small vertical displacement.

- The expression

$$ds = \sqrt{dx^2 + dy^2}$$

is the infinitesimal length of the curve.

- Integrating

$$ds$$

over the interval adds together all of these infinitesimal lengths, producing the total arc length.

Thus,

$$\boxed{L = \int ds.}$$

6.5 Conditions for the Formula

The arc length formula is valid provided that

1. the function

$$f(x)$$

is continuous on the interval,

2. the derivative

$$f'(x)$$

exists,

3. the derivative

$$f'(x)$$

is continuous (or at least piecewise continuous),

4. the integral exists.

These assumptions guarantee that the curve is smooth and that its length can be computed using integration.

6.6 Summary

The derivation of the arc length formula follows four fundamental steps.

Approximate the curve by line segments

$\Downarrow$

Use the Pythagorean Theorem

$\Downarrow$

Replace $\dfrac{\Delta y}{\Delta x}$ by $\dfrac{dy}{dx}$

$\Downarrow$

Take the limit

$\Downarrow$

$$L = \int_a^b \sqrt{1+\left(\frac{dy}{dx}\right)^2}\,dx$$

This derivation explains why every arc length problem ultimately reduces to evaluating a definite integral.

7. The Arc Length Formula

The derivation in the previous section leads to a practical formula for computing the length of a smooth curve. Depending on how the curve is described, there are two equivalent forms of the arc length formula.

If the curve is given as

$$y = f(x),$$

then the integration is performed with respect to x.

If instead the curve is given as

$$x = g(y),$$

then the integration is performed with respect to y.

Choosing the appropriate formula often simplifies the computation considerably.

7.1 Arc Length Formula for $y = f(x)$

Suppose a curve is described by

$$y = f(x), \qquad a \le x \le b,$$

where the derivative

$$f'(x)$$

is continuous on the interval.

The length of the curve between

$$x = a \quad \text{and} \quad x = b$$

is

$$\boxed{L = \int_a^b \sqrt{1 + [f'(x)]^2}\, dx.}$$

Since

$$f'(x) = \frac{dy}{dx},$$

this formula is often written as

$$\boxed{L = \int_a^b \sqrt{1 + \left(\frac{dy}{dx}\right)^2}\, dx.}$$

Understanding the Formula The expression

$$\sqrt{1 + \left(\frac{dy}{dx}\right)^2}$$

is called the **arc length integrand**.
Each part has a geometric meaning.

- The number

  $$1$$

  represents the horizontal contribution.
- The derivative

  $$\frac{dy}{dx}$$

  measures the slope of the curve.
- Squaring removes the sign of the slope.
- The square root comes directly from the Pythagorean Theorem.

Consequently,

$$ds = \sqrt{1 + \left(\frac{dy}{dx}\right)^2}\, dx$$

represents a tiny piece of arc length.

7.2 Arc Length Formula for $x = g(y)$

Sometimes solving for

$$y$$

is difficult, but expressing the curve as

$$x = g(y)$$

is much simpler.

In this situation we integrate with respect to

$$y.$$

Suppose

$$x = g(y), \qquad c \le y \le d,$$

and

$$g'(y)$$

is continuous.
Then the arc length is

$$\boxed{L = \int_c^d \sqrt{1 + [g'(y)]^2}\, dy.}$$

Since

$$g'(y) = \frac{dx}{dy},$$

this may also be written as

$$\boxed{L = \int_c^d \sqrt{1 + \left(\frac{dx}{dy}\right)^2}\, dy.}$$

Why Two Formulas? Both formulas compute exactly the same geometric quantity.
The difference lies only in the variable of integration.
If

$$y$$

is naturally expressed as a function of

$$x,$$

then integrating with respect to

$$x$$

is usually simpler.
On the other hand, if

$$x$$

is naturally expressed as a function of

$$y,$$

then integrating with respect to

$$y$$

often leads to a much easier integral.

7.3 Choosing the Appropriate Formula

When solving an arc length problem, the first step is deciding which formula to use.
The following guidelines are useful.

Curve Given As	Formula to Use
$y = f(x)$	$L = \int_a^b \sqrt{1 + \left(\frac{dy}{dx}\right)^2}\, dx$
$x = g(y)$	$L = \int_c^d \sqrt{1 + \left(\frac{dx}{dy}\right)^2}\, dy$

Whenever possible, choose the form that produces the simplest derivative and the simplest integral.

7.4 Special Cases

Some curves have particularly simple arc length formulas.

Horizontal Line If

$$y = c,$$

then

$$\frac{dy}{dx} = 0,$$

and

$$L = \int_a^b 1\, dx = b - a.$$

This agrees with the ordinary distance formula.

Vertical Line If

$$x = c,$$

then

$$\frac{dx}{dy} = 0,$$

and

$$L = \int_c^d 1\, dy = d - c.$$

Again, this agrees with ordinary geometry.

7.5 Common Mistakes

Students frequently make the following errors when computing arc length.

1. Forgetting to square the derivative.
2. Omitting the square root.

3. Using the formula for $y = f(x)$ when the curve is given more naturally as $x = g(y)$.
4. Forgetting to change the limits of integration after switching variables.
5. Attempting to simplify the integrand before computing the derivative.

7.6 Summary

The two arc length formulas are

$$\boxed{L = \int_a^b \sqrt{1 + \left(\frac{dy}{dx}\right)^2}\, dx,}$$

for curves expressed as

$$y = f(x),$$

and

$$\boxed{L = \int_c^d \sqrt{1 + \left(\frac{dx}{dy}\right)^2}\, dy,}$$

for curves expressed as

$$x = g(y).$$

These formulas are mathematically equivalent and should be viewed as two different ways of computing the same geometric quantity.

8. Differential Arc Length

The arc length formulas introduced in the previous section can be expressed in a much more compact and elegant form using the differential arc length, denoted by

$$ds.$$

Rather than viewing the curve as one entire object, we imagine it as being composed of infinitely many infinitesimally small pieces. Each tiny piece has length

$$ds,$$

and the total length of the curve is obtained by adding together all of these infinitesimal lengths.

8.1 Derivation of the Differential Arc Length

Consider two nearby points on a smooth curve,

$$(x, y)$$

and

$$(x + dx,\ y + dy).$$

Since these two points are extremely close together, the corresponding portion of the curve is nearly a straight line.
Applying the Pythagorean Theorem to the infinitesimal right triangle gives

$$\boxed{ds^2 = dx^2 + dy^2.}$$

Taking the positive square root (since length is always nonnegative) yields

$$\boxed{ds = \sqrt{dx^2 + dy^2}.}$$

This simple expression is called the **differential arc length formula**.

8.2 Expressing ds in Terms of dx

When the curve is given by

$$y = f(x),$$

it is convenient to express every quantity in terms of the variable

$$x.$$

Since

$$dy = \frac{dy}{dx}\, dx,$$

we substitute this into

$$ds = \sqrt{dx^2 + dy^2}.$$

Then

$$\begin{aligned} ds &= \sqrt{dx^2 + \left(\frac{dy}{dx} dx\right)^2} \\ &= \sqrt{dx^2 \left[1 + \left(\frac{dy}{dx}\right)^2\right]} \\ &= \boxed{\sqrt{1 + \left(\frac{dy}{dx}\right)^2}\, dx.} \end{aligned}$$

This is precisely the quantity that appears inside the arc length integral.

8.3 Expressing ds in Terms of dy

If the curve is written as

$$x = g(y),$$

then it is more convenient to work with

$$dy.$$

Since

$$dx = \frac{dx}{dy}\, dy,$$

we substitute into the differential formula.

$$ds = \sqrt{\left(\frac{dx}{dy}dy\right)^2 + dy^2}$$

$$= \sqrt{1 + \left(\frac{dx}{dy}\right)^2}\, dy.$$

Thus,

$$\boxed{ds = \sqrt{1 + \left(\frac{dx}{dy}\right)^2}\, dy.}$$

This expression leads directly to the arc length formula for curves written as

$$x = g(y).$$

8.4 Relationship Between ds and Arc Length

The total length of a curve is obtained by integrating the differential arc length over the interval. Symbolically,

$$\boxed{L = \int ds.}$$

Depending on the variable of integration,

$$\boxed{L = \int_a^b \sqrt{1 + \left(\frac{dy}{dx}\right)^2}\, dx}$$

or

$$\boxed{L = \int_c^d \sqrt{1 + \left(\frac{dx}{dy}\right)^2}\, dy.}$$

Thus, the two arc length formulas are simply different versions of the same fundamental equation.

8.5 Geometric Interpretation

The differential arc length

$$ds$$

represents an infinitesimally small piece of the curve.
Just as

$$dx$$

measures a tiny horizontal displacement and

$$dy$$

measures a tiny vertical displacement,

$$ds$$

measures the tiny distance traveled along the curve itself.
This interpretation is illustrated geometrically by the infinitesimal right triangle

$$\boxed{ds^2 = dx^2 + dy^2.}$$

Although this triangle is infinitely small, it satisfies the same Pythagorean relationship as an ordinary right triangle.

8.6 Why Is ds Important?

The differential arc length appears throughout advanced calculus and mathematical physics.
It is used not only to compute the length of curves but also in formulas for

- surface area of revolution,
- line integrals,
- work done by variable forces,
- mass of wires with variable density,
- center of mass,
- moments of inertia,
- differential geometry.

For this reason,

$$ds$$

is one of the most important differential quantities introduced in Calculus II.

8.7 Summary

The differential arc length provides a compact way of expressing the length of any smooth curve.
The essential formulas are

$$\boxed{ds = \sqrt{dx^2 + dy^2},}$$

$$\boxed{ds = \sqrt{1 + \left(\frac{dy}{dx}\right)^2}\, dx,}$$

and

$$\boxed{ds = \sqrt{1 + \left(\frac{dx}{dy}\right)^2}\, dy.}$$

These formulas form the foundation for both arc length calculations and the surface area formulas developed later in this lecture.

9. The Arc Length Function

In the previous sections, we developed formulas for computing the total length of a curve over a fixed interval. Sometimes, however, we are interested in measuring the distance traveled along a curve from a fixed starting point to an arbitrary point on the curve.
This idea leads naturally to the **arc length function**.

Instead of returning a single number, the arc length function assigns to every point on the curve the distance traveled from the initial point to that location.
Thus, it is an example of an *accumulation function*, similar to the area functions encountered in Calculus I.

9.1 Definition of the Arc Length Function

Suppose

$$y = f(x),$$

where

$$f'(x)$$

is continuous on an interval

$$[a, b].$$

Choose the point

$$(a, f(a))$$

as the starting point.
The arc length measured from

$$x = a$$

to any point

$$x$$

on the curve is denoted by

$$s(x).$$

Definition 9.1. The arc length function is defined by

$$\boxed{s(x) = \int_a^x \sqrt{1 + (f'(t))^2}\, dt.}$$

Notice that the variable of integration is written as

$$t,$$

while

$$x$$

appears only as the upper limit of integration.
This distinction avoids confusion between the integration variable and the evaluation point.

9.2 Geometric Interpretation

The quantity

$$s(x)$$

represents the distance measured along the curve from the initial point

$$(a, f(a))$$

to the point

$$(x, f(x)).$$

Unlike the ordinary distance formula, which measures the straight-line distance between two points, the arc length function measures the distance *along the curve itself.*
Consequently,

$$s(a) = 0,$$

because the distance from the starting point to itself is zero.
As

$$x$$

moves along the curve,

$$s(x)$$

increases continuously.

9.3 Relationship with the Fundamental Theorem of Calculus

The arc length function is an accumulation function, so its derivative can be found using the Fundamental Theorem of Calculus.
Since

$$s(x) = \int_a^x \sqrt{1 + (f'(t))^2}\, dt,$$

the Fundamental Theorem immediately gives

$$\boxed{s'(x) = \sqrt{1 + (f'(x))^2}.}$$

This result has a simple interpretation.
The derivative of the accumulated arc length is precisely the *instantaneous rate at which arc length is increasing.*
Notice that

$$s'(x) = \frac{ds}{dx},$$

so we recover the differential relationship

$$ds = \sqrt{1 + \left(\frac{dy}{dx}\right)^2}\, dx.$$

Thus, the differential arc length introduced in the previous section is completely consistent with the Fundamental Theorem of Calculus.

9.4 Properties of the Arc Length Function

The arc length function possesses several useful properties.

1.
$$s(a) = 0.$$

2.
$$s(x) \geq 0$$

   for every point on the curve.

3. Since

$$\sqrt{1 + (f'(x))^2} > 0,$$

   the function

$$s(x)$$

   is always increasing.

4. The derivative satisfies

$$s'(x) = \sqrt{1 + (f'(x))^2}.$$

5. The total arc length of the curve from

$$a$$

   to

$$b$$

   is simply

$$L = s(b).$$

9.5 Procedure for Finding an Arc Length Function

Whenever an arc length function is requested, the same general procedure is used.

1. Differentiate the given function.
2. Compute

$$1 + (f'(x))^2 .$$

3. Simplify the square root whenever possible.
4. Replace the upper limit by the variable

$$x.$$

5. Evaluate the resulting integral if possible.

Unlike ordinary arc length problems, the final answer is a function rather than a single numerical value.

9.6 Remark

Many arc length integrals cannot be evaluated using elementary functions.
For this reason, textbook examples are carefully chosen so that

$$1 + \left(f'(x)\right)^2$$

simplifies into a perfect square.
Whenever such a simplification occurs, the resulting integral becomes much easier to evaluate.
The example in the next section illustrates this important idea.

9.7 Summary

The arc length function measures the accumulated distance traveled along a curve.
Its defining formula is

$$\boxed{s(x) = \int_a^x \sqrt{1 + \left(f'(t)\right)^2}\, dt,}$$

and its derivative is

$$\boxed{s'(x) = \sqrt{1 + \left(f'(x)\right)^2}.}$$

The total length of the curve is obtained simply by evaluating

$$\boxed{L = s(b).}$$

In the following examples, we apply the arc length formulas and the arc length function to compute the lengths of several curves arising in applications.

10. Examples of Arc Length

We now apply the arc length formulas developed in the previous sections to compute the lengths of several curves.
Each example follows the same general strategy:

1. Rewrite the curve in the appropriate form.
2. Compute the derivative.
3. Substitute into the arc length formula.
4. Simplify the integrand whenever possible.
5. Evaluate the resulting definite integral.

10.1 Example 1

Find the length of the curve

$$y^2 = 4(x+4)^3,$$

for

$$0 \le x \le 2, \qquad y > 0.$$

Step 1. Solve for y Since

$$y > 0,$$

we take the positive square root.

$$\boxed{y = 2(x+4)^{3/2}.}$$

Now the curve is expressed in the form

$$y = f(x),$$

so we use the arc length formula

$$L = \int_a^b \sqrt{1 + \left(\frac{dy}{dx}\right)^2}\, dx.$$

Step 2. Differentiate Differentiate using the Power Rule.

$$\begin{aligned}\frac{dy}{dx} &= 2 \cdot \frac{3}{2}(x+4)^{1/2} \\ &= 3(x+4)^{1/2}.\end{aligned}$$

Thus,

$$\boxed{\frac{dy}{dx} = 3\sqrt{x+4}.}$$

Step 3. Form the Arc Length Integrand Compute

$$1 + \left(\frac{dy}{dx}\right)^2.$$

Since

$$\left(3\sqrt{x+4}\right)^2 = 9(x+4),$$

we obtain

$$\begin{aligned}1 + \left(\frac{dy}{dx}\right)^2 &= 1 + 9(x+4) \\ &= 9x + 37.\end{aligned}$$

Therefore,

$$\sqrt{1 + \left(\frac{dy}{dx}\right)^2} = \sqrt{9x+37}.$$

The arc length integral becomes

$$\boxed{L = \int_0^2 \sqrt{9x+37}\, dx.}$$

Notice that the complicated radical in the original equation has simplified to a very manageable square root.

Step 4. Evaluate the Integral Use the substitution

$$u = 9x + 37.$$

Then

$$du = 9\,dx, \qquad dx = \frac{du}{9}.$$

When

$$x = 0,$$

$$u = 37.$$

When

$$x = 2,$$

$$u = 55.$$

Hence,

$$\begin{aligned} L &= \frac{1}{9}\int_{37}^{55} u^{1/2}\,du \\ &= \frac{1}{9}\left[\frac{2}{3}u^{3/2}\right]_{37}^{55} \\ &= \frac{2}{27}\left[u^{3/2}\right]_{37}^{55}. \end{aligned}$$

Substituting the limits,

$$\boxed{L = \frac{2}{27}\left(55^{3/2} - 37^{3/2}\right).}$$

Since

$$a^{3/2} = a\sqrt{a},$$

the answer may also be written as

$$\boxed{L = \frac{2}{27}\left(55\sqrt{55} - 37\sqrt{37}\right).}$$

Discussion This example illustrates several important features of arc length problems.

- The equation of the curve was first rewritten in explicit form.
- The derivative was relatively simple.
- The expression

$$1 + \left(\frac{dy}{dx}\right)^2$$

simplified to a linear polynomial.

- A straightforward substitution completed the integration.

Many textbook examples are intentionally designed so that the quantity

$$1 + \left(\frac{dy}{dx}\right)^2$$

simplifies nicely. When this does not occur, arc length integrals often cannot be expressed in terms of elementary functions and may require numerical methods.

10.2 Example 2: Arc Length with Respect to y

Find the length of the curve

$$x = \frac{y^4}{8} + \frac{1}{4y^2},$$

for

$$1 \le y \le 2.$$

The curve is already expressed as a function of y. Therefore, it is most natural to use the arc length formula

$$L = \int_c^d \sqrt{1 + \left(\frac{dx}{dy}\right)^2}\, dy.$$

Trying to solve the equation for y as a function of x would make the problem unnecessarily complicated.

Step 1. Rewrite Using Exponents Write the function as

$$x = \frac{1}{8}y^4 + \frac{1}{4}y^{-2}.$$

This form makes differentiation easier.

Step 2. Differentiate with Respect to y Using the Power Rule,

$$\begin{aligned} \frac{dx}{dy} &= \frac{1}{8}(4y^3) + \frac{1}{4}(-2y^{-3}) \\ &= \frac{y^3}{2} - \frac{1}{2y^3}. \end{aligned}$$

Therefore,

$$\boxed{\frac{dx}{dy} = \frac{y^3}{2} - \frac{1}{2y^3}.}$$

Step 3. Compute the Quantity Under the Square Root We need to simplify

$$1 + \left(\frac{dx}{dy}\right)^2.$$

Substitute the derivative:

$$1 + \left(\frac{y^3}{2} - \frac{1}{2y^3}\right)^2.$$

Expand the square carefully:

$$\begin{aligned}\left(\frac{y^3}{2} - \frac{1}{2y^3}\right)^2 &= \frac{y^6}{4} - 2\left(\frac{y^3}{2}\right)\left(\frac{1}{2y^3}\right) + \frac{1}{4y^6} \\ &= \frac{y^6}{4} - \frac{1}{2} + \frac{1}{4y^6}.\end{aligned}$$

Adding 1,

$$\begin{aligned}1 + \left(\frac{dx}{dy}\right)^2 &= 1 + \frac{y^6}{4} - \frac{1}{2} + \frac{1}{4y^6} \\ &= \frac{y^6}{4} + \frac{1}{2} + \frac{1}{4y^6}.\end{aligned}$$

The expression is a perfect square:

$$\frac{y^6}{4} + \frac{1}{2} + \frac{1}{4y^6} = \left(\frac{y^3}{2} + \frac{1}{2y^3}\right)^2.$$

Thus,

$$\boxed{1 + \left(\frac{dx}{dy}\right)^2 = \left(\frac{y^3}{2} + \frac{1}{2y^3}\right)^2.}$$

Step 4. Simplify the Square Root Therefore,

$$\sqrt{1 + \left(\frac{dx}{dy}\right)^2} = \left|\frac{y^3}{2} + \frac{1}{2y^3}\right|.$$

Since

$$1 \le y \le 2,$$

both terms are positive. Hence the absolute value may be removed:

$$\boxed{\sqrt{1 + \left(\frac{dx}{dy}\right)^2} = \frac{y^3}{2} + \frac{1}{2y^3}.}$$

This perfect-square simplification is the key step in the problem.

Step 5. Set Up the Arc Length Integral Using the limits $1 \le y \le 2$,

$$\begin{aligned} L &= \int_1^2 \sqrt{1+\left(\frac{dx}{dy}\right)^2}\,dy \\ &= \int_1^2 \left(\frac{y^3}{2}+\frac{1}{2y^3}\right) dy. \end{aligned}$$

Equivalently,

$$L = \frac{1}{2}\int_1^2 \left(y^3 + y^{-3}\right) dy.$$

Step 6. Evaluate the Integral Integrate term by term:

$$\int \frac{y^3}{2}\,dy = \frac{y^4}{8},$$

and

$$\int \frac{1}{2}y^{-3}\,dy = -\frac{1}{4y^2}.$$

Therefore,

$$L = \left[\frac{y^4}{8} - \frac{1}{4y^2}\right]_1^2.$$

Evaluate at $y = 2$:

$$\frac{2^4}{8} - \frac{1}{4(2^2)} = 2 - \frac{1}{16} = \frac{31}{16}.$$

Evaluate at $y = 1$:

$$\frac{1}{8} - \frac{1}{4} = -\frac{1}{8}.$$

Thus,

$$\begin{aligned} L &= \frac{31}{16} - \left(-\frac{1}{8}\right) \\ &= \frac{31}{16} + \frac{2}{16} \\ &= \frac{33}{16}. \end{aligned}$$

Hence, the length of the curve is

$$\boxed{L = \frac{33}{16}.}$$

Discussion This example illustrates several important ideas.

- Since the curve was given as $x = g(y)$, integration with respect to y was the natural choice.
- The derivative had the form

$$a - b,$$

while the expression under the square root simplified to

$$(a + b)^2.$$

- Recognizing a perfect square eliminated the radical and made the integration elementary.
- The positivity of y on the interval justified removing the absolute value.

A common strategy in arc length problems is to search for algebraic patterns that turn

$$1 + \left(\frac{dx}{dy}\right)^2$$

or

$$1 + \left(\frac{dy}{dx}\right)^2$$

into a perfect square.