# Calculus II

## Lecture 5

## Contents

# 1. Sequences

A sequence is one of the most fundamental objects in calculus. Many important ideas, including limits, infinite series, power series, and Taylor series, are built upon sequences. Before studying infinite sums, we first need to understand how individual terms behave as the index becomes very large.

Intuitively, a sequence is simply an ordered list of numbers. Unlike a set, the order of the numbers matters, and every term has a specific position. Throughout this chapter, we are interested in answering questions such as

- Does the sequence approach a particular number?
- Does it increase or decrease without bound?
- Does it oscillate forever?
- Can we predict its long-term behavior?

Understanding these questions will allow us to determine whether infinite series converge or diverge.

## 1.1 Definition of a Sequence

**Definition 1.1.** A **sequence** is an ordered list of numbers

$$a_1, a_2, a_3, \ldots, a_n, \ldots$$

where

$$a_n$$

denotes the $n^{\text{th}}$ term of the sequence.

The variable $n$ is called the **index** and usually takes values

$$n = 1, 2, 3, \ldots$$

although some sequences begin with $n = 0$.

Unlike a function whose domain may be an interval of real numbers, a sequence is a function whose domain consists of the positive integers.
That is,

$$a : \mathbb{N} \to \mathbb{R},$$

where

$$a(n) = a_n.$$

Consequently, every sequence may be viewed as a function whose graph consists only of isolated points rather than a continuous curve.

### 1.2 Ways to Define a Sequence

There are several common ways to describe a sequence.

**1. Explicit Formula**

Each term is determined directly from the index.

$$a_n = \frac{n}{n+1}.$$

To compute the fifth term,

$$a_5 = \frac{5}{6}.$$

No previous terms are required.

**2. Recursive Formula**

A recursive sequence is defined using one or more previous terms.
For example, the Fibonacci sequence is defined by

$$F_1 = 1, \qquad F_2 = 1,$$

and

$$F_n = F_{n-1} + F_{n-2}, \qquad n \geq 3.$$

Computing the first several terms gives

$$1, 1, 2, 3, 5, 8, 13, \ldots$$

Notice that every new term depends on the two previous terms.

### 1.3 Examples of Sequences

The following examples illustrate different types of sequences.

**Example 1.2.** Find the first five terms of

$$a_n = \frac{n}{n+1}.$$

**Solution**
Substituting $n = 1, 2, 3, 4, 5$ gives

$$\frac{1}{2}, \frac{2}{3}, \frac{3}{4}, \frac{4}{5}, \frac{5}{6}.$$

As $n$ increases, the denominator exceeds the numerator by only one, suggesting that the sequence approaches 1.

**Example 1.3.** Consider

$$a_n = \sqrt{n-3}, \qquad n \geq 3.$$

The first few terms are

$$0,\ 1,\ \sqrt{2},\ \sqrt{3},\ 2, \ldots$$

Since $\sqrt{n}$ grows without bound, the sequence also increases without bound.

**Example 1.4.** Consider

$$a_n = \cos\left(\frac{n\pi}{6}\right).$$

Evaluating several terms,

$$\frac{\sqrt{3}}{2}, \frac{1}{2}, 0, -\frac{1}{2}, -\frac{\sqrt{3}}{2}, -1, -\frac{\sqrt{3}}{2}, \ldots$$

The sequence never approaches a single number. Instead, it oscillates between several values.

**Example 1.5.** Consider

$$\left\{\frac{3}{5}, -\frac{4}{25}, \frac{5}{125}, -\frac{6}{625}, \ldots\right\}.$$

The signs alternate while the denominators grow exponentially.
Consequently, the terms become smaller and smaller in magnitude and appear to approach zero.

**Example 1.6** (Digits of $e$)**.** A sequence does not have to be generated by a formula.
The decimal digits of

$$e = 2.718281828459\cdots$$

produce the sequence

$$7, 1, 8, 2, 8, 1, 8, 2, 8, 4, 5, \ldots$$

Although this is a perfectly valid sequence, it has no obvious limiting behavior.

*Remark.* Whenever possible, it is helpful to compute the first several terms of a sequence before attempting to analyze its limit. Looking at the initial terms often suggests whether the sequence is increasing, decreasing, oscillating, bounded, or approaching a finite value.

## 1.4 Limits of Sequences

Just as we studied limits of functions in Calculus I, we are interested in determining the long-term behavior of sequences. Instead of asking what happens as a variable $x$ approaches a number or infinity, we ask what happens to the terms of a sequence as the index $n$ becomes arbitrarily large.
If the terms become arbitrarily close to a fixed real number, we say the sequence **converges**. Otherwise, the sequence is said to **diverge**.

**Definition**

**Definition 1.7.** A sequence

$$\{a_n\}$$

has limit $L$ if the terms $a_n$ can be made arbitrarily close to $L$ by taking $n$ sufficiently large.
In this case we write

$$\boxed{\lim_{n\to\infty} a_n = L.}$$

If such a number $L$ exists, the sequence is called **convergent**. Otherwise it is called **divergent**.

The phrase "as $n \to \infty$" does not mean that $n$ actually reaches infinity. Rather, it means that we consider larger and larger values of $n$ and observe the behavior of the sequence.

**Visual Interpretation**

Imagine plotting the points

$$(1, a_1), (2, a_2), (3, a_3), \ldots$$

on the coordinate plane.
If the points get closer and closer to the horizontal line

$$y = L,$$

then

$$\lim_{n\to\infty} a_n = L.$$

Unlike functions, the graph of a sequence consists only of isolated points because the domain contains only integers.

## 1.5 Infinite Limits

Sometimes the terms of a sequence increase without bound.

**Definition 1.8.** We write

$$\boxed{\lim_{n\to\infty} a_n = \infty}$$

if for every real number $M$, there exists an integer $N$ such that

$$n > N \quad\Longrightarrow\quad a_n > M.$$

Similarly,

$$\boxed{\lim_{n\to\infty} a_n = -\infty}$$

means the sequence decreases without bound.

Notice that

$$\infty$$

is **not** a real number.
Therefore,

$$\lim_{n\to\infty} a_n = \infty$$

means the sequence diverges, although it diverges in a predictable manner.

## 1.6 Relationship Between Functions and Sequences

Many sequences are obtained by evaluating a function at the positive integers.
For example,

$$f(x) = \frac{x}{x+1}$$

generates the sequence

$$a_n = f(n) = \frac{n}{n+1}.$$

This observation leads to one of the most useful results in this chapter.

**Theorem 1.9** (Limits of Functions and Sequences)**.** *Suppose*

$$a_n = f(n)$$

*for every positive integer $n$.*
*If*

$$\lim_{x\to\infty} f(x) = L,$$

*then*

$$\boxed{\lim_{n\to\infty} a_n = L.}$$

This theorem allows us to evaluate many sequence limits simply by using the limit techniques learned in Calculus I.

## 1.7 Examples

**Example 1.10.** Evaluate

$$\lim_{n\to\infty} \frac{n}{n+1}.$$

**Solution**
Treat the sequence as the function

$$f(x) = \frac{x}{x+1}.$$

Divide numerator and denominator by $x$.

$$\frac{x}{x+1} = \frac{1}{1+\frac{1}{x}}.$$

As

$$x \to \infty,$$

we have

$$\frac{1}{x} \to 0.$$

Therefore,

$$\boxed{\lim_{n\to\infty} \frac{n}{n+1} = 1.}$$

**Example 1.11.** Find

$$\lim_{n\to\infty} \frac{3n^2+5}{7n^2-4n+2}.$$

**Solution**
Divide numerator and denominator by the highest power,

$$n^2.$$

$$= \frac{3+\frac{5}{n^2}}{7-\frac{4}{n}+\frac{2}{n^2}}.$$

As

$$n \to \infty,$$

every fractional term approaches zero.
Hence,

$$\boxed{\lim_{n\to\infty} \frac{3n^2+5}{7n^2-4n+2} = \frac{3}{7}.}$$

**Example 1.12.** Evaluate

$$\lim_{n\to\infty} \frac{1}{\sqrt{n}}.$$

**Solution**
As

$$n \to \infty,$$

the denominator increases without bound.
Therefore,

$$\boxed{\lim_{n\to\infty} \frac{1}{\sqrt{n}} = 0.}$$

More generally,

$$\boxed{\lim_{n\to\infty} \frac{1}{n^r} = 0, \qquad r > 0.}$$

This important limit appears frequently throughout the study of convergence tests.

**Example 1.13.** Evaluate

$$\lim_{n\to\infty} \sqrt{n-3}.$$

**Solution**
As

$$n$$

becomes arbitrarily large,

$$n-3 \to \infty.$$

Taking square roots preserves this behavior.
Therefore,

$$\boxed{\lim_{n\to\infty} \sqrt{n-3} = \infty.}$$

The sequence diverges because it does not approach a finite number.

**Example 1.14.** Determine whether

$$a_n = \cos\left(\frac{n\pi}{6}\right)$$

has a limit.
**Solution**
The first several terms are

$$\frac{\sqrt{3}}{2},\ \frac{1}{2},\ 0,\ -\frac{1}{2},\ -\frac{\sqrt{3}}{2},\ -1, \ldots$$

The values continue repeating forever.
Since the sequence does not approach one number,

$$\boxed{\lim_{n\to\infty} \cos\left(\frac{n\pi}{6}\right) \text{ does not exist.}}$$

The sequence diverges by oscillation.

## 1.8 Important Observations

- Every convergent sequence is bounded.
- A sequence that approaches $\infty$ or $-\infty$ is divergent.
- Oscillating sequences usually diverge because they fail to approach a single real number.
- Whenever possible, rewrite the sequence as a familiar function and use limit techniques from Calculus I.

## 1.9 Limit Laws for Sequences

Fortunately, the limit laws for sequences are almost identical to the limit laws for functions. Once the limits of two sequences are known, limits involving sums, products, quotients, and powers can be evaluated without returning to the definition of a limit.

**Theorem 1.15** (Limit Laws)**.** *Suppose*

$$\lim_{n\to\infty} a_n = A, \qquad \lim_{n\to\infty} b_n = B,$$

*and let $c$ be a constant. Then*

1. $$\lim_{n\to\infty}(a_n+b_n)=A+B.$$

2. $$\lim_{n\to\infty}(a_n-b_n)=A-B.$$

3. $$\lim_{n\to\infty}(ca_n)=cA.$$

4. $$\lim_{n\to\infty}(a_nb_n)=AB.$$

5. $$\lim_{n\to\infty}\frac{a_n}{b_n}=\frac{A}{B},\qquad B\neq 0.$$

6. $$\lim_{n\to\infty}(a_n)^p=A^p,$$

   *provided $p>0$ and $a_n>0$ eventually.*

These rules allow us to compute complicated limits by evaluating each piece separately.

### 1.10 Examples Using the Limit Laws

**Example 1.16.** Evaluate

$$\lim_{n\to\infty}\left(\frac{2n}{n+1}+\frac{5}{n}\right).$$

**Solution**
Using the limit laws,

$$=2\lim_{n\to\infty}\frac{n}{n+1}+5\lim_{n\to\infty}\frac{1}{n}.$$

Since

$$\lim_{n\to\infty}\frac{n}{n+1}=1,$$

and

$$\lim_{n\to\infty}\frac{1}{n}=0,$$

we obtain

$$\boxed{2(1)+5(0)=2.}$$

**Example 1.17.** Evaluate

$$\lim_{n\to\infty}\frac{4n^2+3n-1}{2n^2+n+5}.$$

**Solution**
Divide numerator and denominator by $n^2$.

$$= \frac{4 + \frac{3}{n} - \frac{1}{n^2}}{2 + \frac{1}{n} + \frac{5}{n^2}}.$$

Taking limits,

$$= \frac{4}{2} = 2.$$

Therefore,

$$\boxed{\lim_{n\to\infty} \frac{4n^2 + 3n - 1}{2n^2 + n + 5} = 2.}$$

### 1.11 The Squeeze Theorem

Sometimes it is difficult to compute a limit directly. Instead, we compare the sequence with two simpler sequences whose limits are already known.
This idea leads to one of the most useful tools in calculus.

**Theorem 1.18** (Squeeze Theorem)**.** *Suppose that*

$$a_n \le b_n \le c_n$$

*for all sufficiently large values of $n$.*
*If*

$$\lim_{n\to\infty} a_n = \lim_{n\to\infty} c_n = L,$$

*then*

$$\boxed{\lim_{n\to\infty} b_n = L.}$$

The middle sequence is "squeezed" between two sequences having the same limit, forcing it to approach that same value.

### 1.12 Example: Using the Squeeze Theorem

**Example 1.19.** Evaluate

$$\lim_{n\to\infty} \frac{\sin n}{n}.$$

**Solution**
Since

$$-1 \le \sin n \le 1,$$

dividing by the positive number $n$ gives

$$-\frac{1}{n} \le \frac{\sin n}{n} \le \frac{1}{n}.$$

Taking limits,

$$\lim_{n\to\infty} -\frac{1}{n} = 0,$$

and

$$\lim_{n\to\infty} \frac{1}{n} = 0.$$

Therefore,

$$\boxed{\lim_{n\to\infty} \frac{\sin n}{n} = 0.}$$

### 1.13 Monotone Sequences

Many sequences continually increase or continually decrease.
Such sequences have especially nice convergence properties.

**Definition 1.20.** A sequence

$$\{a_n\}$$

is called

- **Increasing** if

  $$a_{n+1} > a_n$$

  for every $n$.
- **Decreasing** if

  $$a_{n+1} < a_n$$

  for every $n$.
- **Monotone** if it is either increasing or decreasing.

### 1.14 Bounded Sequences

**Definition 1.21.** A sequence is
**bounded above** if there exists a number $M$ such that

$$a_n \leq M$$

for every $n$.
Similarly, it is
**bounded below**
if there exists a number $m$ satisfying

$$a_n \geq m$$

for every $n$.
If both conditions hold, the sequence is said to be **bounded**.

### 1.15 Monotone Convergence Theorem

One of the most important results concerning sequences is the following.

**Theorem 1.22** (Monotone Convergence Theorem)**.**

1. *Every increasing sequence that is bounded above converges.*
2. *Every decreasing sequence that is bounded below converges.*

This theorem is extremely important because many sequences arising in calculus are monotone. Instead of computing the limit directly, we can often prove that the sequence is monotone and bounded, guaranteeing the existence of a limit.

### 1.16 Examples

**Example 1.23.** Determine whether

$$a_n = 1 - \frac{1}{n}$$

is increasing.

**Solution**

Compute

$$a_{n+1} = 1 - \frac{1}{n+1}.$$

Since

$$\frac{1}{n+1} < \frac{1}{n},$$

we have

$$1 - \frac{1}{n+1} > 1 - \frac{1}{n}.$$

Therefore,

$$\boxed{a_{n+1} > a_n,}$$

so the sequence is increasing.

Furthermore,

$$a_n < 1$$

for every $n$, so it is bounded above.

Hence the sequence converges.

Indeed,

$$\boxed{\lim_{n\to\infty} \left(1 - \frac{1}{n}\right) = 1.}$$

## 2. Infinite Series

In the previous section, we studied sequences, which are simply ordered lists of numbers. We now consider what happens when we *add* the terms of a sequence.

Infinite series arise naturally throughout mathematics, physics, engineering, economics, and statistics. They allow us to represent functions, approximate complicated quantities, solve differential equations, and compute numerical values with remarkable accuracy.

Unlike a finite sum, an infinite series contains infinitely many terms. Since it is impossible to literally add infinitely many numbers, we instead study the sequence of *partial sums*. The behavior of these partial sums determines whether the series converges or diverges.

### 2.1 Definition of an Infinite Series

**Definition 2.1.** Let

$$\{a_n\}$$

be a sequence.
The expression

$$\boxed{\sum_{n=1}^{\infty} a_n = a_1 + a_2 + a_3 + \cdots}$$

is called an **infinite series**.
Each $a_n$ is called a **term** of the series.

Notice that a sequence is simply a list of numbers, whereas a series is the sum of those numbers.
For example,

$$1,\ \frac{1}{2},\ \frac{1}{4},\ \frac{1}{8}, \ldots$$

is a sequence, while

$$1 + \frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \cdots$$

is an infinite series.

### 2.2 Sigma Notation

Most infinite series are written using sigma notation.

$$\boxed{\sum_{n=1}^{\infty} a_n}$$

where

- $\sum$ (capital Greek letter Sigma) means "add",
- $n = 1$ is the starting index,
- $\infty$ indicates that the process continues forever,
- $a_n$ is the general term.

For example,

$$\sum_{n=1}^{5} n = 1 + 2 + 3 + 4 + 5.$$

Likewise,

$$\sum_{n=0}^{4} \left(\frac{1}{2}\right)^n = 1 + \frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \frac{1}{16}.$$

### 2.3 Partial Sums

Since an infinite number of additions cannot actually be performed, we instead consider finite sums.

**Definition 2.2.** The $n^{\mathbf{th}}$ **partial sum** of a series is

$$\boxed{S_n = a_1 + a_2 + \cdots + a_n.}$$

The sequence

$$S_1, S_2, S_3, \ldots$$

is called the sequence of partial sums.

The key idea is that an infinite series is completely determined by the behavior of its partial sums.

## 2.4 Convergence of a Series

**Definition 2.3.** An infinite series

$$\sum_{n=1}^{\infty} a_n$$

is said to **converge** if the sequence of partial sums converges.
That is,

$$\boxed{\lim_{n\to\infty} S_n = L.}$$

The number $L$ is called the **sum** of the series.
If the limit does not exist, then the series is said to **diverge**.

Notice something important.
We are *not* asking whether

$$a_n$$

has a limit.
Instead, we are asking whether

$$S_n$$

has a limit.
These are completely different questions.

## 2.5 Example 1

**Example 2.4.** Consider the series

$$1 + \frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \cdots$$

Find several partial sums.
**Solution**

$$S_1 = 1$$

$$S_2 = 1 + \frac{1}{2} = \frac{3}{2}$$

$$S_3 = 1 + \frac{1}{2} + \frac{1}{4} = \frac{7}{4}$$

$$S_4 = \frac{15}{8}$$

$$S_5 = \frac{31}{16}$$

The partial sums become

$$1, 1.5, 1.75, 1.875, 1.9375, \ldots$$

These values approach

$$2.$$

Therefore,

$$\boxed{1 + \frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \cdots = 2.}$$

This is one of the most important infinite series in mathematics.

### 2.6 Example 2

**Example 2.5.** Determine whether

$$1 + 2 + 3 + 4 + \cdots$$

converges.

**Solution**

The partial sums are

$$1, 3, 6, 10, 15, 21, \ldots$$

Since

$$S_n = \frac{n(n+1)}{2},$$

we have

$$S_n \to \infty.$$

Therefore,

$$\boxed{1 + 2 + 3 + 4 + \cdots \text{ diverges.}}$$

### 2.7 Important Remark

A common misconception is that the terms of a series become very small, so the series must converge.
This is false.
Even though

$$\frac{1}{n} \to 0,$$

the harmonic series

$$\sum_{n=1}^{\infty} \frac{1}{n}$$

actually diverges.

We will prove this later using the Integral Test.
Therefore,

$$a_n \to 0$$

is necessary for convergence, but it is not sufficient.

## 2.8 Arithmetic Operations on Series

Suppose

$$\sum a_n$$

and

$$\sum b_n$$

are convergent series.
Then

$$\sum (a_n + b_n) = \sum a_n + \sum b_n.$$

Similarly,

$$\sum (a_n - b_n) = \sum a_n - \sum b_n.$$

Finally,

$$\sum (ca_n) = c \sum a_n,$$

where $c$ is a constant.
These properties follow directly from the corresponding properties of limits applied to partial sums.

## 2.9 Geometric Series

One of the simplest and most useful infinite series is the **geometric series**. Geometric series appear in finance, computer science, engineering, probability, physics, and many areas of applied mathematics. Unlike most infinite series, geometric series can be summed exactly.

### Definition

**Definition 2.6.** A **geometric sequence** is a sequence in which every term is obtained by multiplying the previous term by a fixed constant $r$, called the **common ratio**.
If the first term is $a$, then the sequence has the form

$$a,\ ar,\ ar^2,\ ar^3,\ \dots$$

The corresponding infinite series is

$$\boxed{\sum_{n=0}^{\infty} ar^n = a + ar + ar^2 + ar^3 + \cdots .}$$

The value of $r$ determines whether the terms increase, decrease, alternate in sign, or remain constant.

## 2.10 Examples of Geometric Sequences

**Example 2.7.** Determine the first five terms of the geometric sequence with

$$a = 3, \qquad r = 2.$$

**Solution**
Multiply each term by 2.

$$3,\ 6,\ 12,\ 24,\ 48,\dots$$

The sequence increases rapidly because the common ratio is greater than one.

**Example 2.8.** Find the first five terms if

$$a = 5, \qquad r = \frac{1}{3}.$$

$$5,\ \frac{5}{3},\ \frac{5}{9},\ \frac{5}{27},\ \frac{5}{81},\dots$$

Each term becomes smaller since

$$0 < r < 1.$$

**Example 2.9.** Find the first six terms of

$$a = 2, \qquad r = -\frac{1}{2}.$$

$$2, -1, \frac{1}{2}, -\frac{1}{4}, \frac{1}{8}, -\frac{1}{16}, \dots$$

The terms alternate signs because the common ratio is negative.

## 2.11 Finite Geometric Sum

To understand infinite geometric series, we first derive a formula for the sum of the first $n$ terms. Consider

$$S_n = a + ar + ar^2 + \cdots + ar^{n-1}.$$

Multiply both sides by $r$.

$$rS_n = ar + ar^2 + \cdots + ar^n.$$

Subtract the two equations.

$$S_n - rS_n = a - ar^n.$$

Factor.

$$(1 - r)S_n = a(1 - r^n).$$

Therefore,

$$\boxed{S_n = \frac{a(1 - r^n)}{1 - r}, \qquad r \neq 1.}$$

This is called the **finite geometric series formula**.

## 2.12 Infinite Geometric Series

Now let

$$n \to \infty.$$

If

$$|r| < 1,$$

then

$$r^n \to 0.$$

Substituting into the finite sum formula,

$$S = \frac{a}{1-r}.$$

This gives the fundamental theorem.

**Theorem 2.10** (Infinite Geometric Series)**.** *The geometric series*

$$\sum_{n=0}^{\infty} ar^n$$

*converges if and only if*

$$\boxed{|r| < 1.}$$

*When it converges,*

$$\boxed{\sum_{n=0}^{\infty} ar^n = \frac{a}{1-r}.}$$

*If*

$$|r| \geq 1,$$

*the series diverges.*

## 2.13 Why Does the Condition $|r| < 1$ Matter?

The key observation is that the formula

$$\frac{a}{1-r}$$

is valid only when

$$r^n \to 0.$$

This happens exactly when

$$|r| < 1.$$

If

$$r = 1,$$

the terms never become smaller.
If

$$r > 1,$$

the terms grow larger and larger.
If

$$r = -1,$$

the partial sums oscillate forever.
Thus the only possibility for convergence is

$$|r| < 1.$$

### 2.14 Examples

**Example 2.11.** Evaluate

$$1 + \frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \cdots .$$

**Solution**
Here,

$$a = 1, \qquad r = \frac{1}{2}.$$

Since

$$\left|\frac{1}{2}\right| < 1,$$

the series converges.
Applying the formula,

$$\sum_{n=0}^{\infty} \left(\frac{1}{2}\right)^n = \frac{1}{1 - \frac{1}{2}} = 2.$$

$$\boxed{2}$$

**Example 2.12.** Determine whether

$$3 + 6 + 12 + 24 + \cdots$$

converges.
**Solution**
The common ratio is

$$r = 2.$$

Since

$$|2| > 1,$$

the geometric series diverges.

$$\boxed{\text{Diverges}}$$

**Example 2.13.** Evaluate

$$\sum_{n=0}^{\infty} 5\left(-\frac{1}{3}\right)^n.$$

**Solution**
Here,

$$a = 5, \qquad r = -\frac{1}{3}.$$

Since

$$\left|-\frac{1}{3}\right| < 1,$$

the series converges.
Therefore,

$$= \frac{5}{1+\frac{1}{3}} = \frac{5}{\frac{4}{3}} = \frac{15}{4}.$$

$$\boxed{\frac{15}{4}}$$

**Example 2.14.** Determine whether

$$\sum_{n=1}^{\infty} \left(\frac{4}{5}\right)^n$$

converges.
**Solution**
Notice that the first term is

$$a = \frac{4}{5}.$$

The common ratio is also

$$r = \frac{4}{5}.$$

Since

$$|r| < 1,$$

the series converges.
Using the geometric formula,

$$S = \frac{\frac{4}{5}}{1-\frac{4}{5}} = \frac{\frac{4}{5}}{\frac{1}{5}} = 4.$$

Therefore,

$$\boxed{4.}$$

## 2.15 Recognizing Geometric Series

A series is geometric if the ratio of consecutive terms is constant.

That is,

$$\frac{a_{n+1}}{a_n} = r$$

for every $n$.
For example,

$$1 + \frac{1}{3} + \frac{1}{9} + \frac{1}{27} + \cdots$$

is geometric because

$$\frac{\frac{1}{3}}{1} = \frac{\frac{1}{9}}{\frac{1}{3}} = \frac{1}{3}.$$

### 2.16 Common Mistakes

Students often make the following mistakes.

1. Using the formula

$$\frac{a}{1-r}$$

without first checking that

$$|r| < 1.$$

2. Using the wrong first term. Always identify the first term exactly as written.
3. Confusing the first term with the common ratio.
4. Forgetting that geometric series beginning with $n = 1$ may have a different first term than those beginning with $n = 0$.

### 2.17 Telescoping Series

Not every infinite series requires a special convergence test. Some series simplify naturally because most of their terms cancel when the partial sums are written explicitly. Such series are called **telescoping series**.

**Definition**

**Definition 2.15.** An infinite series is called a **telescoping series** if, after writing the partial sums,

$$S_n = a_1 + a_2 + \cdots + a_n,$$

most terms cancel, leaving only a few remaining terms.

The idea is similar to collapsing a telescope: many intermediate pieces slide together, leaving only the beginning and the end.

### 2.18 Example 1

**Example 2.16.** Determine whether

$$\sum_{n=1}^{\infty} \left( \frac{1}{n} - \frac{1}{n+1} \right)$$

converges.

**Solution**

Compute the first several partial sums.

$$S_1 = 1 - \frac{1}{2}.$$

$$S_2 = \left(1 - \frac{1}{2}\right) + \left(\frac{1}{2} - \frac{1}{3}\right).$$

Notice that

$$-\frac{1}{2} + \frac{1}{2}$$

cancel.
Continuing,

$$S_3 = \left(1 - \frac{1}{2}\right) + \left(\frac{1}{2} - \frac{1}{3}\right) + \left(\frac{1}{3} - \frac{1}{4}\right).$$

Again,

$$-\frac{1}{3} + \frac{1}{3}$$

cancel.
Therefore,

$$S_n = 1 - \frac{1}{n+1}.$$

Taking the limit,

$$\lim_{n\to\infty} S_n = 1 - 0 = 1.$$

Hence,

$$\boxed{\sum_{n=1}^{\infty} \left(\frac{1}{n} - \frac{1}{n+1}\right) = 1.}$$

### 2.19 Example 2

**Example 2.17.** Evaluate

$$\sum_{n=1}^{\infty} \frac{2}{n(n+2)}.$$

**Solution**
First perform partial fraction decomposition.
Assume

$$\frac{2}{n(n+2)} = \frac{A}{n} + \frac{B}{n+2}.$$

Multiplying through,

$$2 = A(n+2) + Bn.$$

Comparing coefficients,

$$A = 1, \qquad B = -1.$$

Thus,

$$\frac{2}{n(n+2)} = \frac{1}{n} - \frac{1}{n+2}.$$

The series becomes

$$\sum_{n=1}^{\infty} \left( \frac{1}{n} - \frac{1}{n+2} \right).$$

Writing the first several terms,

$$\left(1 - \frac{1}{3}\right) + \left(\frac{1}{2} - \frac{1}{4}\right) + \left(\frac{1}{3} - \frac{1}{5}\right) + \cdots$$

Most terms cancel.
The partial sums become

$$S_n = 1 + \frac{1}{2} - \frac{1}{n+1} - \frac{1}{n+2}.$$

Taking limits,

$$\boxed{\sum_{n=1}^{\infty} \frac{2}{n(n+2)} = \frac{3}{2}.}$$

### 2.20 How to Recognize a Telescoping Series

A telescoping series often contains expressions such as

$$\frac{1}{n} - \frac{1}{n+1},$$

or

$$f(n) - f(n+1),$$

where consecutive terms naturally cancel.
Many rational functions become telescoping only after performing partial fraction decomposition.

### 2.21 The Divergence Test

The divergence test is usually the first test that should be applied to an unfamiliar series.
Unlike most convergence tests, this test cannot prove that a series converges. It can only prove that a series diverges.

**Theorem**

**Theorem 2.18** (Divergence Test)**.** *If*

$$\sum_{n=1}^{\infty} a_n$$

*converges, then*

$$\boxed{\lim_{n\to\infty} a_n = 0.}$$

*Equivalently,*
*if*

$$\lim_{n\to\infty} a_n \neq 0,$$

*or if the limit does not exist,*
*then*

$$\boxed{\sum_{n=1}^{\infty} a_n \text{ diverges.}}$$

### 2.22 Why Does This Work?

Suppose

$$S_n = a_1 + a_2 + \cdots + a_n$$

converges to some number $L$.
Then

$$S_{n-1}$$

also approaches $L$.
Since

$$a_n = S_n - S_{n-1},$$

we have

$$a_n \longrightarrow L - L = 0.$$

Thus, every convergent series must have terms approaching zero.
The converse, however, is false.

### 2.23 Example 1

**Example 2.19.** Determine whether

$$\sum_{n=1}^{\infty} \frac{n}{n+1}$$

converges.
**Solution**
Compute the limit of the terms.

$$\lim_{n\to\infty} \frac{n}{n+1} = 1.$$

Since

$$1 \neq 0,$$

the Divergence Test tells us immediately that

$$\boxed{\sum_{n=1}^{\infty} \frac{n}{n+1} \text{ diverges.}}$$

### 2.24 Example 2

**Example 2.20.** Determine whether

$$\sum_{n=1}^{\infty} (-1)^n$$

converges.

**Solution**

The terms are

$$-1, 1, -1, 1, \ldots$$

The limit does not exist because the terms oscillate.

Therefore,

$$\boxed{\sum_{n=1}^{\infty} (-1)^n \text{ diverges.}}$$

### 2.25 Example 3

**Example 2.21.** Can the Divergence Test determine whether

$$\sum_{n=1}^{\infty} \frac{1}{n}$$

converges?

**Solution**

We compute

$$\lim_{n \to \infty} \frac{1}{n} = 0.$$

Since the limit equals zero, the Divergence Test gives no conclusion.

The test is *inconclusive*.

In fact, the harmonic series diverges, but a different convergence test (such as the Integral Test) is required to prove it.

## 3. The Integral Test

The Divergence Test tells us when a series definitely diverges, but it cannot prove that a series converges. To analyze many positive-term series, we introduce one of the most important convergence tests in Calculus II: the **Integral Test**.

The main idea is simple. If the terms of a series come from a positive, continuous, decreasing function, then the behavior of the series is closely related to the behavior of an improper integral.

### 3.1 Motivation

Suppose we have the series

$$\sum_{n=1}^{\infty} a_n,$$

where

$$a_n = f(n)$$

for some function $f(x)$.
If $f(x)$ is positive and decreasing, then the rectangles of width 1 and height $f(n)$ closely approximate the area under the curve $y = f(x)$.
This observation suggests that the convergence of the infinite series should be related to the convergence of the improper integral

$$\int_1^\infty f(x)\,dx.$$

The Integral Test makes this relationship precise.

### 3.2 Statement of the Integral Test

**Theorem 3.1** (Integral Test)**.** *Suppose*

$$a_n = f(n),$$

*where the function $f(x)$ satisfies the following conditions on $[1, \infty)$:*

1. *$f(x)$ is continuous,*
2. *$f(x) > 0$,*
3. *$f(x)$ is decreasing.*

*Then*

$$\boxed{\sum_{n=1}^\infty a_n}$$

*and*

$$\boxed{\int_1^\infty f(x)\,dx}$$

*either both converge or both diverge.*

### 3.3 Important Observations

Notice what the theorem does **not** say.
It does not say that the value of the series equals the value of the integral.
Instead, it says that they have the same convergence behavior.
That is,

$$\boxed{\begin{aligned} \int_1^\infty f(x)\,dx \text{ converges} &\Longrightarrow \sum_{n=1}^\infty a_n \text{ converges,} \\ \int_1^\infty f(x)\,dx \text{ diverges} &\Longrightarrow \sum_{n=1}^\infty a_n \text{ diverges.} \end{aligned}}$$

### 3.4 Checklist Before Using the Integral Test

Before applying the Integral Test, always verify the following.

$$\boxed{\begin{array}{ll} \checkmark & f(x) \text{ is continuous,} \\ \checkmark & f(x) > 0, \\ \checkmark & f(x) \text{ is decreasing,} \\ \checkmark & a_n = f(n). \end{array}}$$

If any of these conditions fail, the Integral Test cannot be applied.

### 3.5 Example 1: The Series $\sum \frac{\ln n}{n}$

**Example 3.2.** Determine whether

$$\sum_{n=2}^{\infty} \frac{\ln n}{n}$$

converges.

**Solution**

Let

$$f(x) = \frac{\ln x}{x}.$$

First verify the hypotheses.

Since

$$x > 1,$$

the function is continuous and positive.

Differentiate.

$$f'(x) = \frac{1 - \ln x}{x^2}.$$

Notice that

$$f'(x) < 0$$

whenever

$$\ln x > 1,$$

or equivalently,

$$x > e.$$

Thus $f(x)$ is eventually decreasing, which is sufficient for the Integral Test.

Now evaluate the improper integral.

Let

$$u = \ln x, \qquad du = \frac{1}{x}\,dx.$$

Then

$$\int_2^{\infty} \frac{\ln x}{x}\,dx = \lim_{b\to\infty} \int_{\ln 2}^{\ln b} u\,du.$$

Integrating,

$$= \lim_{b\to\infty} \frac{u^2}{2}\Bigg|_{\ln 2}^{\ln b}.$$

Therefore,

$$= \lim_{b\to\infty} \frac{(\ln b)^2}{2} - \frac{(\ln 2)^2}{2}.$$

Since

$$(\ln b)^2 \to \infty,$$

the improper integral diverges.
Hence,

$$\boxed{\sum_{n=2}^{\infty} \frac{\ln n}{n} \text{ diverges.}}$$

### 3.6 Example 2: The Series $\sum \frac{\ln n}{n^2}$

**Example 3.3.** Determine whether

$$\sum_{n=2}^{\infty} \frac{\ln n}{n^2}$$

converges.
**Solution**
Let

$$f(x) = \frac{\ln x}{x^2}.$$

The function is continuous and positive for $x > 1$.
Differentiate.

$$f'(x) = \frac{1 - 2\ln x}{x^3}.$$

For sufficiently large $x$,

$$1 - 2\ln x < 0,$$

so the function is decreasing.
Apply the Integral Test.
Use integration by parts.
Choose

$$u = \ln x, \qquad dv = \frac{1}{x^2}\,dx.$$

Then

$$du = \frac{1}{x}\,dx, \qquad v = -\frac{1}{x}.$$

Therefore,

$$\int \frac{\ln x}{x^2}\,dx = -\frac{\ln x}{x} + \int \frac{1}{x^2}\,dx.$$

Thus,

$$= -\frac{\ln x}{x} - \frac{1}{x}.$$

Evaluating the improper integral,

$$\lim_{b\to\infty}\left(-\frac{\ln b}{b} - \frac{1}{b}\right) - \left(-\frac{\ln 2}{2} - \frac{1}{2}\right).$$

Both fractions involving $b$ approach zero.
Hence the improper integral converges.
Therefore,

$$\boxed{\sum_{n=2}^{\infty} \frac{\ln n}{n^2} \text{ converges.}}$$

### 3.7 When Should You Use the Integral Test?

The Integral Test is especially useful for series involving

$$\frac{1}{n}, \qquad \frac{1}{n^p}, \qquad \frac{\ln n}{n}, \qquad \frac{\ln n}{n^2}, \qquad \frac{1}{n(\ln n)^p},$$

or other expressions that are easy to integrate.

## 4. The $p$-Series Test

Among all infinite series, the family

$$\sum_{n=1}^{\infty} \frac{1}{n^p}$$

plays a central role in the study of convergence. These series are called **$p$-series**. They appear frequently throughout calculus and are often used as comparison series in later convergence tests.
Unlike many other infinite series, the convergence of a $p$-series depends entirely on the value of the exponent $p$.

### 4.1 Definition

**Definition 4.1.** A series of the form

$$\boxed{\sum_{n=1}^{\infty} \frac{1}{n^p},}$$

where $p$ is a real number, is called a **$p$-series**.

Some common examples are

$$\sum \frac{1}{n}, \qquad \sum \frac{1}{n^2}, \qquad \sum \frac{1}{n^3}, \qquad \sum \frac{1}{\sqrt{n}}, \qquad \sum \frac{1}{n^{3/2}}.$$

Although these series look similar, they do not all have the same convergence behavior.

### 4.2 The $p$-Series Theorem

**Theorem 4.2** ($p$-Series Test)**.** *The series*

$$\sum_{n=1}^{\infty} \frac{1}{n^p}$$

*satisfies the following:*

$$\boxed{\begin{array}{l} p > 1 \Longrightarrow \textit{Series converges,} \\[1ex] p \leq 1 \Longrightarrow \textit{Series diverges.} \end{array}}$$

This theorem is one of the most frequently used convergence results in Calculus II.

### 4.3 Why is the Critical Value $p = 1$?

The proof follows directly from the Integral Test.
Let

$$f(x) = \frac{1}{x^p}.$$

Since

$$f(x)$$

is continuous, positive, and decreasing on

$$[1, \infty),$$

we examine

$$\int_1^{\infty} \frac{1}{x^p}\, dx.$$

**Case 1:** $p \neq 1$
Recall

$$\int x^{-p}\, dx = \frac{x^{1-p}}{1-p}.$$

Therefore,

$$\int_1^{\infty} \frac{1}{x^p}\, dx = \lim_{b\to\infty} \frac{x^{1-p}}{1-p}\Big|_1^b.$$

There are two possibilities.

If

$$p > 1,$$

then

$$1 - p < 0.$$

Consequently,

$$b^{1-p} \to 0,$$

and the improper integral converges.
Therefore,

$$\boxed{\sum \frac{1}{n^p} \text{ converges whenever } p > 1.}$$

If

$$0 < p < 1,$$

then

$$1 - p > 0.$$

Hence,

$$b^{1-p} \to \infty,$$

so the improper integral diverges.
Therefore,

$$\boxed{\sum \frac{1}{n^p} \text{ diverges whenever } 0 < p < 1.}$$

**Case 2:** $p = 1$
When

$$p = 1,$$

the integral becomes

$$\int_1^\infty \frac{1}{x}\, dx.$$

Since

$$\int \frac{1}{x}\, dx = \ln x,$$

we obtain

$$\lim_{b \to \infty} \ln b = \infty.$$

Therefore,

$$\boxed{\sum_{n=1}^{\infty} \frac{1}{n} \text{ diverges.}}$$

This famous series is called the **harmonic series**.

## 4.4 Summary Table

| $p$ | Behavior |
|---|---|
| $p > 1$ | Converges |
| $p = 1$ | Diverges |
| $0 < p < 1$ | Diverges |
| $p < 0$ | Diverges |

Notice that when

$$p < 0,$$

the terms actually increase instead of approaching zero.

## 4.5 Examples

**Example 4.3.** Determine whether

$$\sum_{n=1}^{\infty} \frac{1}{n^2}$$

converges.

**Solution**

Here,

$$p = 2.$$

Since

$$2 > 1,$$

the series converges by the $p$-Series Test.

$$\boxed{\text{Convergent}}$$

**Example 4.4.** Determine whether

$$\sum_{n=1}^{\infty} \frac{1}{\sqrt{n}}$$

converges.

**Solution**

Rewrite

$$\frac{1}{\sqrt{n}} = \frac{1}{n^{1/2}}.$$

Thus,

$$p = \frac{1}{2}.$$

Since

$$\frac{1}{2} < 1,$$

the series diverges.

$$\boxed{\text{Divergent}}$$

**Example 4.5.** Determine whether

$$\sum_{n=1}^{\infty} \frac{1}{n^{5/3}}$$

converges.
**Solution**
Since

$$p = \frac{5}{3} > 1,$$

the series converges.

$$\boxed{\text{Convergent}}$$

**Example 4.6.** Determine whether

$$\sum_{n=1}^{\infty} \frac{1}{n}$$

converges.
**Solution**
This is the harmonic series.
Since

$$p = 1,$$

the series diverges.

$$\boxed{\text{Divergent}}$$

**Example 4.7.** Determine whether

$$\sum_{n=1}^{\infty} \frac{1}{n^{12}}$$

converges.
**Solution**
Here,

$$p = 12 > 1.$$

Therefore,

$$\boxed{\text{Convergent.}}$$

Notice that larger values of $p$ cause the terms to decrease much more rapidly.

### 4.6 How to Recognize a $p$-Series

Before applying the $p$-Series Test, rewrite the general term in the form

$$\frac{1}{n^p}.$$

For example,

$$\frac{1}{\sqrt{n}} = \frac{1}{n^{1/2}},$$

$$\frac{1}{n\sqrt{n}} = \frac{1}{n^{3/2}},$$

$$\frac{1}{n^4} = \frac{1}{n^4}.$$

If the denominator contains additional factors such as

$$\ln n, \qquad n+1, \qquad \sin n,$$

then the series is generally not a $p$-series, and another test must be used.