Calculus II

Lecture 6

Contents

1. Comparison Tests

Many series are too complicated to analyze directly using the Integral Test or the p-Series Test. Instead, we compare them with a simpler series whose convergence behavior is already known.
The basic philosophy is straightforward:

- If a positive series is *smaller* than a convergent series, then it must also converge.
- If a positive series is *larger* than a divergent series, then it must also diverge.

This idea forms the basis of the Direct Comparison Test.

1.1 Known Benchmark Series

Before applying any comparison test, it is useful to remember several important benchmark series.

Series	Behavior
$\sum \frac{1}{n}$	Diverges
$\sum \frac{1}{n^p},\ p>1$	Converges
$\sum ar^n,\ \lvert r\rvert<1$	Converges
$\sum ar^n,\ \lvert r\rvert\geq 1$	Diverges

These are the series we most often compare with.

1.2 The Direct Comparison Test

Theorem 1.1 (Direct Comparison Test)**.** *Suppose*

$$0 \leq a_n \leq b_n$$

for all sufficiently large n.
Then

1. *If*

$$\sum b_n$$

   converges,

   then

$$\boxed{\sum a_n \textit{ also converges.}}$$

2. *If*

$$a_n \geq b_n \geq 0$$

   and

$$\sum b_n$$

   diverges,

   then

$$\boxed{\sum a_n \textit{ also diverges.}}$$

1.3 Why Does the Test Work?

Imagine adding together positive numbers.
If every term of one series is smaller than the corresponding term of a series that already has a finite sum, then the smaller series cannot possibly exceed that finite value.
Likewise, if every term is larger than those of a divergent series, then its partial sums must also become arbitrarily large.

1.4 Choosing a Comparison

The most important step is selecting an appropriate comparison series.

Expression	Compare With
$\frac{1}{n^2+5}$	$\frac{1}{n^2}$
$\frac{3n+1}{n^3+7}$	$\frac{1}{n^2}$
$\frac{1}{n\sqrt{n}}$	$\frac{1}{n^{3/2}}$
$\frac{5}{2^n+1}$	$\frac{1}{2^n}$

Generally, keep only the dominant term in the numerator and denominator.

1.5 Example 1

Example 1.2. Determine whether

$$\sum_{n=1}^{\infty} \frac{1}{n^2+5}$$

converges.

Solution

Since

$$n^2 + 5 > n^2,$$

we have

$$\frac{1}{n^2+5} < \frac{1}{n^2}.$$

Now,

$$\sum \frac{1}{n^2}$$

is a p-series with

$$p = 2 > 1,$$

so it converges.

Therefore,

$$\boxed{\sum_{n=1}^{\infty} \frac{1}{n^2+5} \text{ converges.}}$$

1.6 Example 2

Example 1.3. Determine whether

$$\sum_{n=1}^{\infty} \frac{1}{\sqrt{n}}$$

converges.

Solution

Observe that

$$\sqrt{n} < n,$$

so

$$\frac{1}{\sqrt{n}} > \frac{1}{n}.$$

Since

$$\sum \frac{1}{n}$$

diverges,

the Direct Comparison Test gives

$$\boxed{\sum \frac{1}{\sqrt{n}} \text{ diverges.}}$$

1.7 Limit Comparison Test

Sometimes it is difficult to prove one series is always larger or smaller than another.
Instead of comparing term-by-term, we compare their long-term behavior.

Theorem 1.4 (Limit Comparison Test)**.** *Suppose*

$$a_n > 0, \qquad b_n > 0,$$

and

$$L = \lim_{n\to\infty} \frac{a_n}{b_n}.$$

If

$$0 < L < \infty,$$

then

$$\boxed{\sum a_n \textit{ and } \sum b_n \textit{ either both converge or both diverge.}}$$

1.8 Why Does This Work?

If

$$\frac{a_n}{b_n} \to L,$$

where

$$L$$

is a positive finite number,
then eventually

$$a_n \approx Lb_n.$$

Thus the two series have essentially the same long-term behavior.

1.9 Example 3

Example 1.5. Determine whether

$$\sum_{n=1}^{\infty} \frac{3n+2}{n^2+5}$$

converges.
Solution
The dominant terms are

$$3n$$

and

$$n^2.$$

Therefore compare with

$$\frac{1}{n}.$$

Compute

$$L = \lim_{n\to\infty} \frac{\frac{3n+2}{n^2+5}}{\frac{1}{n}}.$$

Simplifying,

$$= \lim_{n\to\infty} \frac{3n^2+2n}{n^2+5} = 3.$$

Since

$$0 < 3 < \infty,$$

both series behave identically.
Because

$$\sum \frac{1}{n}$$

diverges,
we conclude

$$\boxed{\sum_{n=1}^{\infty} \frac{3n+2}{n^2+5} \text{ diverges.}}$$

1.10 Example 4

Example 1.6. Determine whether

$$\sum_{n=1}^{\infty} \frac{5n+1}{n^3+4}$$

converges.
Solution
Compare with

$$\frac{1}{n^2}.$$

Compute

$$L = \lim_{n\to\infty} \frac{\frac{5n+1}{n^3+4}}{\frac{1}{n^2}} = \lim_{n\to\infty} \frac{5n^3+n^2}{n^3+4} = 5.$$

Since

$$0 < 5 < \infty,$$

both series have the same behavior.
Because

$$\sum \frac{1}{n^2}$$

converges,
the given series also converges.

$$\boxed{\text{Convergent}}$$

1.11 Direct Comparison vs. Limit Comparison

Use the Direct Comparison Test when an obvious inequality exists.
Examples include

$$\frac{1}{n^2+1} < \frac{1}{n^2},$$

or

$$\frac{1}{\sqrt{n}} > \frac{1}{n}.$$

Use the Limit Comparison Test when the dominant terms determine the behavior but establishing inequalities is inconvenient.

2. Alternating Series

So far we have studied series whose terms are positive. We now consider series whose terms alternate between positive and negative values.
Alternating series arise naturally in many applications, especially in power series and Taylor series, which we will study later.

2.1 Definition

Definition 2.1. An **alternating series** is an infinite series whose consecutive terms alternate in sign. Typical forms include

$$\sum_{n=1}^{\infty} (-1)^{n-1} a_n$$

or

$$\sum_{n=1}^{\infty} (-1)^{n} a_n,$$

where

$$a_n > 0$$

for every n.

For example,

$$1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \frac{1}{5} - \cdots$$

and

$$1 - \frac{1}{4} + \frac{1}{9} - \frac{1}{16} + \cdots$$

are alternating series.

2.2 Behavior of Partial Sums

Unlike positive-term series, the partial sums of an alternating series typically oscillate around the actual sum.
For example,

$$1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \cdots$$

has partial sums

$$S_1 = 1,$$

$$S_2 = \frac{1}{2},$$

$$S_3 = \frac{5}{6},$$

$$S_4 = \frac{7}{12},$$

and so on.
Notice that the partial sums alternately overestimate and underestimate the true value of the series.

2.3 Alternating Series Test

Theorem 2.2 (Alternating Series Test)**.** *Consider the alternating series*

$$\sum_{n=1}^{\infty} (-1)^{n-1} a_n,$$

where

$$a_n > 0.$$

If

1. *the sequence*

$$a_n$$

is decreasing,

$$a_{n+1} \le a_n,$$

and

2. $$\lim_{n\to\infty} a_n = 0,$$

then the alternating series converges.

2.4 Checklist for Applying the Test

Before applying the Alternating Series Test, verify all of the following.

✓	Signs alternate.
✓	$a_n > 0$.
✓	a_n decreases.
✓	$a_n \to 0$.

If any condition fails, the Alternating Series Test cannot be applied.

2.5 Why Does the Test Work?

As the terms decrease in magnitude, each successive positive or negative term becomes smaller than the previous correction.
Consequently,

- the odd partial sums decrease,
- the even partial sums increase,
- both sequences approach the same limiting value.

Thus the series converges.

2.6 Example 1

Example 2.3. Determine whether

$$\sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n}$$

converges.
Solution
Write

$$a_n = \frac{1}{n}.$$

We verify the conditions.

$$a_n > 0,$$

$$a_n = \frac{1}{n}$$

is decreasing,
and

$$\lim_{n\to\infty} \frac{1}{n} = 0.$$

Therefore, the Alternating Series Test applies.
Hence,

$$\boxed{\sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n} \text{ converges.}}$$

2.7 Example 2

Example 2.4. Determine whether

$$\sum_{n=1}^{\infty}(-1)^n\frac{1}{\sqrt{n}}$$

converges.

Solution

Let

$$a_n = \frac{1}{\sqrt{n}}.$$

Since

$$a_n > 0,$$

$$a_n$$

is decreasing,

and

$$\lim_{n\to\infty}\frac{1}{\sqrt{n}} = 0,$$

the Alternating Series Test guarantees convergence.

$$\boxed{\text{Convergent}}$$

2.8 Example 3

Example 2.5. Determine whether

$$\sum_{n=1}^{\infty}(-1)^n$$

converges.

Solution

Here,

$$a_n = 1.$$

Although the signs alternate,

$$\lim_{n\to\infty} a_n = 1 \neq 0.$$

Therefore,

the Alternating Series Test does not apply.

The Divergence Test immediately gives

$$\boxed{\text{The series diverges.}}$$

2.9 Alternating Series Estimation Theorem

One advantage of alternating series is that we can estimate the error made by using a partial sum.

Theorem 2.6 (Alternating Series Estimation)**.** *Suppose*

$$\sum_{n=1}^{\infty}(-1)^{n-1}a_n$$

satisfies the hypotheses of the Alternating Series Test.
If

$$S$$

is the exact sum and

$$S_n$$

is the nth partial sum, then

$$\boxed{|S - S_n| < a_{n+1}.}$$

In words, the error is no larger than the first omitted term.

2.10 Example 4

Example 2.7. Approximate

$$\sum_{n=1}^{\infty}\frac{(-1)^{n-1}}{n}$$

using

$$S_4.$$

Estimate the maximum possible error.
Solution
Compute

$$S_4 = 1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} = \frac{7}{12}.$$

The next omitted term is

$$a_5 = \frac{1}{5}.$$

Therefore,

$$\boxed{|S - S_4| < \frac{1}{5}.}$$

Thus

$$S \approx \frac{7}{12},$$

with an error less than

$$\frac{1}{5}.$$

2.11 How Many Terms Are Needed?

Suppose we want the approximation error to be less than a prescribed tolerance ε.
Since

$$|S - S_n| < a_{n+1},$$

we simply choose

$$n$$

large enough so that

$$a_{n+1} < \varepsilon.$$

3. Absolute and Conditional Convergence

In the previous section, we learned that many alternating series converge because the positive and negative terms partially cancel each other.
This raises an important question.

> Would the series still converge if all of the negative signs were removed?

The answer leads to two fundamental concepts:

- Absolute convergence
- Conditional convergence

Absolute convergence is one of the strongest forms of convergence for infinite series.

3.1 Absolute Convergence

Definition 3.1. A series

$$\sum_{n=1}^{\infty} a_n$$

is said to converge **absolutely** if

$$\boxed{\sum_{n=1}^{\infty} |a_n|}$$

converges.

In other words, ignore all signs and determine whether the resulting positive series converges.
If it does, then the original series converges absolutely.

3.2 Conditional Convergence

Definition 3.2. A series

$$\sum a_n$$

is said to converge **conditionally** if

1. the original series converges,
2. but

$$\sum |a_n|$$

diverges.

Conditional convergence occurs because the alternating signs create enough cancellation to produce convergence, even though the corresponding positive series diverges.

3.3 Relationship Between the Two

The following diagram summarizes the possibilities.

$$\boxed{\begin{array}{c} \sum |a_n| \text{ converges} \\ \Downarrow \\ \sum a_n \text{ converges} \end{array}}$$

Thus,

$$\boxed{\text{Absolute convergence} \implies \text{Convergence.}}$$

However,

$$\boxed{\text{Convergence} \not\Rightarrow \text{Absolute convergence.}}$$

Some convergent series are only conditionally convergent.

3.4 The Absolute Convergence Theorem

Theorem 3.3. *If*

$$\sum_{n=1}^{\infty} |a_n|$$

converges, then

$$\boxed{\sum_{n=1}^{\infty} a_n}$$

also converges.

Notice that the converse is not always true.

3.5 Example 1: Absolute Convergence

Example 3.4. Determine whether

$$\sum_{n=1}^{\infty} \frac{(-1)^n}{n^2}$$

converges absolutely, conditionally, or diverges.

Solution

First examine

$$\sum \left| \frac{(-1)^n}{n^2} \right| = \sum \frac{1}{n^2}.$$

This is a p-series with

$$p = 2 > 1,$$

so it converges.
Therefore,

$$\boxed{\text{The series converges absolutely.}}$$

Since absolute convergence implies convergence, no further work is required.

3.6 Example 2: Conditional Convergence

Example 3.5. Determine whether

$$\sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n}$$

converges absolutely or conditionally.
Solution
First consider the absolute value.

$$\sum \left| \frac{(-1)^{n-1}}{n} \right| = \sum \frac{1}{n}.$$

This is the harmonic series, which diverges.
Now examine the original series.
Since

$$\frac{1}{n}$$

is positive,
decreasing,
and approaches zero,
the Alternating Series Test shows that

$$\sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n}$$

converges.
Therefore,

$$\boxed{\text{The series converges conditionally.}}$$

3.7 Example 3

Example 3.6. Determine whether

$$\sum_{n=1}^{\infty} (-1)^n \frac{1}{\sqrt{n}}$$

is absolutely or conditionally convergent.
Solution
First remove the alternating sign.

$$\sum \frac{1}{\sqrt{n}} = \sum \frac{1}{n^{1/2}}.$$

Since

$$p = \frac{1}{2} < 1,$$

this p-series diverges.
Now apply the Alternating Series Test to the original series.
The sequence

$$a_n = \frac{1}{\sqrt{n}}$$

is positive,
decreasing,
and satisfies

$$a_n \to 0.$$

Hence,
the original series converges.
Therefore,

$$\boxed{\text{The series is conditionally convergent.}}$$

3.8 Example 4

Example 3.7. Determine whether

$$\sum_{n=1}^{\infty} (-1)^n$$

is absolutely convergent, conditionally convergent, or divergent.
Solution
Since

$$\lim_{n\to\infty} (-1)^n$$

does not exist,
the Divergence Test shows immediately that the series diverges.
Therefore,

$$\boxed{\text{The series diverges.}}$$

3.9 A General Strategy

Whenever a series contains alternating signs, use the following procedure.

Step 1.
Ignore the alternating signs and determine whether

$$\sum |a_n|$$

converges.

- If it converges, stop.

  The series is absolutely convergent.

Step 2.
If

$$\sum |a_n|$$

diverges, test the original series using the Alternating Series Test.

- If the original series converges,

  the convergence is conditional.

- Otherwise,

  the series diverges.

3.10 Common Mistakes

1. Forgetting to remove the alternating sign before testing for absolute convergence.
2. Declaring a series conditionally convergent without first showing that the corresponding absolute-value series diverges.
3. Assuming every alternating series is conditionally convergent.

   Many alternating series actually converge absolutely.
4. Forgetting that absolute convergence automatically guarantees ordinary convergence.

Summary

For an alternating series

$$\sum a_n,$$

always begin by considering

$$\sum |a_n|.$$

$\sum \|a_n\|$	Conclusion
Converges	Absolutely convergent
Diverges	Apply the Alternating Series Test

If the original alternating series converges while the absolute-value series diverges, then the series is

$$\boxed{\text{conditionally convergent.}}$$

4. The Ratio Test

Many infinite series contain factorials, exponential functions, or products that make comparison tests difficult to apply. In these cases, the Ratio Test often provides the quickest method for determining convergence.

The basic idea is to compare consecutive terms of the series. If the terms decrease rapidly enough from one term to the next, the series will converge. If they fail to decrease sufficiently—or even increase—the series will diverge.

4.1 Motivation

Suppose

$$\sum_{n=1}^{\infty} a_n$$

is a series with positive terms.
Instead of studying the terms themselves, consider the ratio

$$\frac{a_{n+1}}{a_n}.$$

This ratio measures how quickly consecutive terms change.

- If the ratio is much less than 1, each term is significantly smaller than the previous one, suggesting convergence.
- If the ratio is greater than 1, the terms are increasing, so the series cannot converge.
- If the ratio approaches 1, the test cannot distinguish between convergent and divergent behavior.

4.2 The Ratio Test

Theorem 4.1 (Ratio Test)**.** *Let*

$$\sum_{n=1}^{\infty} a_n$$

be a series with

$$a_n \neq 0$$

for sufficiently large n.
Suppose

$$L = \lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right|.$$

Then

$$\boxed{\begin{aligned} L<1 &\implies \textit{Series converges absolutely.} \\ L>1 &\implies \textit{Series diverges.} \\ L=1 &\implies \textit{The test is inconclusive.} \end{aligned}}$$

4.3 Why Absolute Values?

Notice that the Ratio Test uses

$$\left|\frac{a_{n+1}}{a_n}\right|.$$

The absolute value ignores alternating signs and measures only the size of consecutive terms. Consequently, whenever the Ratio Test shows convergence, it actually proves

$$\boxed{\text{absolute convergence.}}$$

4.4 Procedure for Using the Ratio Test

To apply the Ratio Test, follow these steps.

1. Write the general term
$$a_n.$$
2. Compute
$$a_{n+1}.$$
3. Form
$$\left|\frac{a_{n+1}}{a_n}\right|.$$
4. Simplify completely.
5. Evaluate
$$L = \lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right|.$$
6. Compare L with 1.

4.5 Example 1: A Factorial Series

Example 4.2. Determine whether

$$\sum_{n=1}^{\infty}\frac{n!}{5^n}$$

converges.

Solution

Let

$$a_n = \frac{n!}{5^n}.$$

Then

$$a_{n+1} = \frac{(n+1)!}{5^{\,n+1}}.$$

Compute the ratio.

$$\left|\frac{a_{n+1}}{a_n}\right| = \frac{(n+1)!}{5^{\,n+1}}\cdot\frac{5^n}{n!}.$$

Simplify.

$$= \frac{n+1}{5}.$$

Now compute the limit.

$$L = \lim_{n\to\infty}\frac{n+1}{5} = \infty.$$

Since

$$L > 1,$$

the series diverges.

$$\boxed{\text{Divergent}}$$

4.6 Example 2: An Exponential Series

Example 4.3. Determine whether

$$\sum_{n=0}^{\infty} \frac{3^n}{n!}$$

converges.

Solution

Let

$$a_n = \frac{3^n}{n!}.$$

Then

$$a_{n+1} = \frac{3^{n+1}}{(n+1)!}.$$

Therefore,

$$\left| \frac{a_{n+1}}{a_n} \right| = \frac{3^{n+1}}{(n+1)!} \cdot \frac{n!}{3^n} = \frac{3}{n+1}.$$

Taking the limit,

$$L = \lim_{n\to\infty} \frac{3}{n+1} = 0.$$

Since

$$L < 1,$$

the series converges absolutely.

$$\boxed{\text{Absolutely convergent}}$$

4.7 Example 3: A Power Series

Example 4.4. Determine whether

$$\sum_{n=1}^{\infty} \frac{n}{2^n}$$

converges.

Solution

Let

$$a_n = \frac{n}{2^n}.$$

Then

$$a_{n+1} = \frac{n+1}{2^{n+1}}.$$

Thus,

$$\left|\frac{a_{n+1}}{a_n}\right| = \frac{n+1}{2^{n+1}} \cdot \frac{2^n}{n} = \frac{n+1}{2n}.$$

Therefore,

$$L = \lim_{n\to\infty} \frac{n+1}{2n} = \frac{1}{2}.$$

Since

$$\frac{1}{2} < 1,$$

the series converges absolutely.

$$\boxed{\text{Convergent}}$$

4.8 Example 4: An Inconclusive Case

Example 4.5. Apply the Ratio Test to

$$\sum_{n=1}^{\infty} \frac{1}{n}.$$

Solution
Let

$$a_n = \frac{1}{n}.$$

Then

$$\left|\frac{a_{n+1}}{a_n}\right| = \frac{n}{n+1}.$$

Hence,

$$L = \lim_{n\to\infty} \frac{n}{n+1} = 1.$$

Since

$$L = 1,$$

the Ratio Test gives no conclusion.
Indeed, the harmonic series diverges, but another test (such as the Integral Test or the p-Series Test) must be used.

4.9 When Should You Use the Ratio Test?

The Ratio Test is particularly effective for series involving

$$n!, \qquad (2n)!, \qquad a^n, \qquad e^n, \qquad n^n,$$

or products of factorials and exponentials.
It is often the first choice whenever factorials appear.

5. The Root Test

The Root Test is another powerful convergence test for infinite series. Instead of comparing consecutive terms, it examines the long-term behavior of the nth root of each term.
The test is especially useful when the general term contains an expression raised to the power n, such as

$$\left(\frac{2n}{3n+1}\right)^n, \qquad \left(\frac{4}{5}\right)^n, \qquad \left(\frac{n}{n+1}\right)^n.$$

In many such cases, taking the nth root greatly simplifies the expression.

5.1 The Root Test

Theorem 5.1 (Root Test)**.** *Let*

$$\sum_{n=1}^{\infty} a_n$$

be an infinite series.
Suppose

$$L = \lim_{n\to\infty} \sqrt[n]{|a_n|}.$$

Then

$$\boxed{\begin{aligned} L<1 &\implies \text{The series converges absolutely.}\\ L>1 &\implies \text{The series diverges.}\\ L=1 &\implies \text{The test is inconclusive.}\end{aligned}}$$

Notice that the conclusions are identical to those of the Ratio Test.

5.2 Why Does the Root Test Work?

Suppose

$$\sqrt[n]{|a_n|} \approx L.$$

Then

$$|a_n| \approx L^n.$$

Thus, for large values of n, the given series behaves like the geometric series

$$\sum L^n.$$

Recall that a geometric series converges when

$$|L| < 1$$

and diverges when

$$|L| > 1.$$

This observation explains the conclusions of the Root Test.

5.3 Procedure for Applying the Root Test

To apply the Root Test, follow these steps.

1. Write the general term
$$a_n.$$
2. Compute
$$\sqrt[n]{|a_n|}.$$
3. Simplify as much as possible.
4. Compute
$$L = \lim_{n\to\infty} \sqrt[n]{|a_n|}.$$
5. Compare L with 1.

5.4 Example 1

Example 5.2. Determine whether

$$\sum_{n=1}^{\infty} \left(\frac{3}{4}\right)^n$$

converges.

Solution

Let

$$a_n = \left(\frac{3}{4}\right)^n.$$

Then

$$\sqrt[n]{|a_n|} = \frac{3}{4}.$$

Hence,

$$L = \frac{3}{4}.$$

Since

$$\frac{3}{4} < 1,$$

the Root Test shows that the series converges absolutely.

$$\boxed{\text{Convergent}}$$

5.5 Example 2

Example 5.3. Determine whether

$$\sum_{n=1}^{\infty} \left(\frac{2n}{3n+1} \right)^n$$

converges.
Solution
Compute

$$\sqrt[n]{|a_n|} = \frac{2n}{3n+1}.$$

Now evaluate the limit.

$$L = \lim_{n\to\infty} \frac{2n}{3n+1} = \frac{2}{3}.$$

Since

$$\frac{2}{3} < 1,$$

the series converges absolutely.

$$\boxed{\text{Absolutely convergent}}$$

5.6 Example 3

Example 5.4. Determine whether

$$\sum_{n=1}^{\infty} \left(\frac{5n}{2n+1} \right)^n$$

converges.
Solution
Compute

$$\sqrt[n]{|a_n|} = \frac{5n}{2n+1}.$$

Therefore,

$$L = \lim_{n\to\infty} \frac{5n}{2n+1} = \frac{5}{2}.$$

Since

$$\frac{5}{2} > 1,$$

the series diverges.

$$\boxed{\text{Divergent}}$$

5.7 Example 4

Example 5.5. Apply the Root Test to

$$\sum_{n=1}^{\infty} \frac{1}{n}.$$

Solution
Here,

$$a_n = \frac{1}{n}.$$

Compute

$$\sqrt[n]{a_n} = n^{-1/n}.$$

Using the standard limit

$$\lim_{n\to\infty} n^{1/n} = 1,$$

we obtain

$$L = 1.$$

Therefore,
the Root Test is inconclusive.
Indeed, the harmonic series diverges, but another test is required to show this.

5.8 Ratio Test vs. Root Test

Both tests have the same conclusions, but they are useful in different situations.

Ratio Test	**Root Test**
Factorials	Terms raised to the nth power
Exponentials	$(a_n)^n$
Products	Power series

Whenever the expression already contains a power of n, the Root Test is usually simpler.

6. Power Series

A polynomial contains only finitely many terms. For example,

$$1 + x + x^2 + x^3$$

is a polynomial of degree three.
A natural question is:

> What happens if we allow infinitely many powers of x?

The answer leads to one of the most important ideas in mathematics—the **power series**. Power series provide a way to represent functions using infinite polynomials and form the foundation of Taylor series, Fourier analysis, differential equations, numerical analysis, and many applications in science and engineering.

6.1 Definition of a Power Series

Definition 6.1. A **power series centered at** c is an infinite series of the form

$$\boxed{\sum_{n=0}^{\infty} a_n (x-c)^n,}$$

where

- a_n are constants called the **coefficients**,
- c is the **center** of the series,
- x is the variable.

When

$$c = 0,$$

the power series becomes

$$\boxed{\sum_{n=0}^{\infty} a_n x^n,}$$

which is called a power series centered at the origin.

6.2 Examples of Power Series

Example 6.2. The series

$$1 + x + x^2 + x^3 + x^4 + \cdots$$

may be written as

$$\sum_{n=0}^{\infty} x^n.$$

Here,

$$a_n = 1, \qquad c = 0.$$

Example 6.3. The series

$$2 + 3(x-1) + 5(x-1)^2 - (x-1)^3 + \cdots$$

is centered at

$$c = 1.$$

Its coefficients are

$$a_0 = 2,\ a_1 = 3,\ a_2 = 5,\ a_3 = -1, \ldots$$

Example 6.4. The series

$$\sum_{n=0}^{\infty} \frac{x^n}{n!}$$

is centered at

$$0.$$

The coefficients are

$$a_n = \frac{1}{n!}.$$

This series will later be shown to represent the exponential function e^x.

6.3 Viewing a Power Series as a Function

Unlike an ordinary numerical series, a power series depends on the variable x.
For some values of x, the series converges.
For other values of x, the series diverges.
Therefore, every power series defines a function only on the set of values of x for which the series converges.
Finding this set is one of the principal goals of this chapter.

6.4 Convergence Depends on x

Consider

$$\sum_{n=0}^{\infty} x^n.$$

This is a geometric series with

$$r = x.$$

From the geometric series theorem,

$$\sum_{n=0}^{\infty} x^n = \frac{1}{1-x},$$

provided

$$|x| < 1.$$

If

$$|x| \geq 1,$$

the geometric series diverges.
Thus,

x	Behavior
$\lvert x\rvert < 1$	Converges
$\lvert x\rvert \geq 1$	Diverges

This simple example illustrates an important fact:

> A power series may converge for some values of x and diverge for others.

6.5 Radius and Interval of Convergence

For every power series

$$\sum_{n=0}^{\infty} a_n (x-c)^n,$$

there exists a nonnegative number R, called the **radius of convergence**, with one of the following possibilities.

1. The series converges whenever

$$|x-c| < R.$$

2. The series diverges whenever

$$|x-c| > R.$$

3. At the endpoints

$$x = c - R \quad \text{and} \quad x = c + R,$$

   the behavior must be checked separately.

The interval consisting of all values of x for which the series converges is called the **interval of convergence**.

6.6 Three Possible Cases

There are only three possibilities.

$R = 0$	Converges only at $x = c$,
$0 < R < \infty$	Converges on a finite interval, with endpoints checked separately,
$R = \infty$	Converges for every real number x.

6.7 Example

Example 6.5. Find the interval of convergence of

$$\sum_{n=0}^{\infty} x^n.$$

Solution

This is a geometric series with ratio

$$r = x.$$

The geometric series converges whenever

$$|x| < 1.$$

Therefore,

$$R = 1.$$

The endpoints are

$$x = -1 \quad \text{and} \quad x = 1.$$

At

$$x = 1,$$

the series becomes

$$1 + 1 + 1 + \cdots,$$

which diverges.

At

$$x = -1,$$

the series becomes

$$1 - 1 + 1 - 1 + \cdots,$$

whose terms do not approach zero, so it also diverges.
Hence,

$$\boxed{\text{Interval of convergence } = (-1, 1).}$$

6.8 General Procedure for Finding an Interval of Convergence

For most power series, use the following procedure.

1. Apply the Ratio Test (or Root Test).
2. Solve the resulting inequality for x.
3. Identify the radius of convergence R.
4. Test each endpoint separately using an appropriate convergence test.
5. Write the interval of convergence using interval notation.

7. Finding the Radius and Interval of Convergence

In the previous section, we learned that every power series has a radius of convergence and an interval of convergence.
The next question is:

> How do we actually determine them?

The answer is that we almost always use the **Ratio Test** (or occasionally the Root Test).
The Ratio Test determines where the series converges absolutely. However, it does *not* determine what happens at the endpoints. Those must always be tested separately.

7.1 General Procedure

Given a power series

$$\sum_{n=0}^{\infty} a_n (x - c)^n,$$

use the following steps.

1. Apply the Ratio Test.
2. Compute

$$L = \lim_{n\to\infty} \left| \frac{a_{n+1}(x - c)^{n+1}}{a_n (x - c)^n} \right|.$$

3. Simplify the expression until only

$$|x - c|$$

remains.

4. Solve

$$L < 1.$$

   This determines the radius of convergence.

5. Test each endpoint separately.
6. Write the interval of convergence.

7.2 Example 1

Example 7.1. Find the radius and interval of convergence of

$$\sum_{n=1}^{\infty} \frac{x^n}{n}.$$

Solution
Let

$$a_n = \frac{x^n}{n}.$$

Apply the Ratio Test.

$$\left| \frac{a_{n+1}}{a_n} \right| = \left| \frac{x^{n+1}}{n+1} \cdot \frac{n}{x^n} \right|.$$

Simplifying,

$$= |x| \cdot \frac{n}{n+1}.$$

Taking the limit,

$$L = |x|.$$

For convergence,

$$|x| < 1.$$

Therefore,

$$\boxed{R = 1.}$$

Now test the endpoints.

Endpoint $x = 1$:
The series becomes

$$\sum_{n=1}^{\infty} \frac{1}{n},$$

which is the harmonic series.
Therefore,

$$\boxed{\text{Diverges.}}$$

Endpoint $x = -1$:
The series becomes

$$\sum_{n=1}^{\infty} \frac{(-1)^n}{n}.$$

This converges by the Alternating Series Test.
Hence,

$$\boxed{\text{Interval of convergence } [-1, 1).}$$

7.3 Example 2

Example 7.2. Determine the radius and interval of convergence of

$$\sum_{n=0}^{\infty} \frac{x^n}{n!}.$$

Solution
Apply the Ratio Test.

$$\left|\frac{a_{n+1}}{a_n}\right| = \left|\frac{x^{n+1}}{(n+1)!} \cdot \frac{n!}{x^n}\right|.$$

Simplifying,

$$= \frac{|x|}{n+1}.$$

Taking the limit,

$$L = 0.$$

Since

$$0 < 1$$

for every real value of x,
the series converges for every real number.
Thus,

$$\boxed{R = \infty.}$$

and

$$\boxed{(-\infty, \infty).}$$

7.4 Example 3

Example 7.3. Find the radius and interval of convergence of

$$\sum_{n=0}^{\infty} n!\, x^n.$$

Solution
Apply the Ratio Test.

$$\left|\frac{a_{n+1}}{a_n}\right| = \left|\frac{(n+1)!x^{n+1}}{n!x^n}\right| = (n+1)|x|.$$

Taking limits,

$$L = \begin{cases} 0, & x = 0, \\ \infty, & x \neq 0. \end{cases}$$

Therefore,
the series converges only when

$$x = 0.$$

Hence,

$$\boxed{R = 0.}$$

The interval of convergence consists of the single point

$$\boxed{\{0\}.}$$

7.5 Example 4

Example 7.4. Determine the radius and interval of convergence of

$$\sum_{n=1}^{\infty} \frac{(x-2)^n}{3^n}.$$

Solution
Apply the Ratio Test.

$$\left|\frac{a_{n+1}}{a_n}\right| = \left|\frac{(x-2)^{n+1}}{3^{n+1}} \cdot \frac{3^n}{(x-2)^n}\right| = \frac{|x-2|}{3}.$$

For convergence,

$$\frac{|x-2|}{3} < 1.$$

Thus,

$$|x-2| < 3.$$

Therefore,

$$\boxed{R = 3.}$$

The possible interval is

$$-1 < x < 5.$$

Now check each endpoint.

Endpoint $x = -1$:
The series becomes

$$\sum (-1)^n,$$

which diverges.

Endpoint $x = 5$:
The series becomes

$$\sum 1,$$

which also diverges.
Hence,

$$\boxed{(-1, 5).}$$

7.6 Important Remarks

Notice that the Ratio Test always determines the radius of convergence.
However, when

$$L = 1,$$

the Ratio Test is inconclusive.
At the endpoints,

$$x = c - R \quad \text{and} \quad x = c + R,$$

the series may

- converge at both endpoints,
- diverge at both endpoints,
- converge at one endpoint and diverge at the other.

Each endpoint must therefore be tested independently using one of the previous convergence tests.

7.7 Common Endpoint Tests

When testing endpoints, the resulting series is often one of the following.

$\sum \frac{1}{n}$	Harmonic Series (diverges)
$\sum \frac{(-1)^n}{n}$	Alternating Harmonic Series (converges)
$\sum \frac{1}{n^p}$	p-Series Test
$\sum ar^n$	Geometric Series

8. Differentiation and Integration of Power Series

One of the remarkable properties of power series is that they behave much like polynomials.
Recall that ordinary polynomials may be differentiated and integrated term-by-term. Surprisingly, the same is true for power series within their interval of convergence.
This allows us to generate new power series from known ones and is one of the primary tools used to derive Taylor and Maclaurin series.

8.1 Term-by-Term Differentiation

Consider the power series

$$f(x)=\sum_{n=0}^{\infty} a_n(x-c)^n.$$

Within its interval of convergence,

$$f(x)=a_0+a_1(x-c)+a_2(x-c)^2+\cdots.$$

Differentiating each term gives

$$f'(x)=a_1+2a_2(x-c)+3a_3(x-c)^2+\cdots.$$

In sigma notation,

$$\boxed{f'(x)=\sum_{n=1}^{\infty} na_n(x-c)^{n-1}.}$$

8.2 Term-by-Term Integration

Similarly,

$$f(x)=\sum_{n=0}^{\infty} a_n(x-c)^n$$

may be integrated term-by-term.
Integrating each term,

$$\int f(x)\,dx=C+a_0(x-c)+\frac{a_1}{2}(x-c)^2+\frac{a_2}{3}(x-c)^3+\cdots.$$

Therefore,

$$\boxed{\int f(x)\,dx=C+\sum_{n=0}^{\infty}\frac{a_n}{n+1}(x-c)^{n+1}.}$$

Notice that an arbitrary constant of integration must always be included.

8.3 Radius of Convergence

An important fact is that differentiation and integration do *not* change the radius of convergence.

Theorem 8.1. *Suppose*

$$\sum_{n=0}^{\infty} a_n(x-c)^n$$

has radius of convergence

$$R.$$

Then both

$$\sum_{n=1}^{\infty} na_n(x-c)^{n-1}$$

and

$$\sum_{n=0}^{\infty} \frac{a_n}{n+1}(x-c)^{n+1}$$

have the same radius of convergence

$$\boxed{R.}$$

However, the behavior at the endpoints may change and must be checked separately.

8.4 Example 1: Differentiating a Geometric Series

Example 8.2. Start with the geometric series

$$\frac{1}{1-x} = \sum_{n=0}^{\infty} x^n, \qquad |x| < 1.$$

Differentiate both sides.

Solution

Differentiating the left-hand side,

$$\frac{d}{dx}\left(\frac{1}{1-x}\right) = \frac{1}{(1-x)^2}.$$

Differentiating the series term-by-term,

$$\frac{d}{dx}\left(\sum_{n=0}^{\infty} x^n\right) = \sum_{n=1}^{\infty} nx^{n-1}.$$

Hence,

$$\boxed{\frac{1}{(1-x)^2} = \sum_{n=1}^{\infty} nx^{n-1}, \qquad |x| < 1.}$$

8.5 Example 2: Integrating a Geometric Series

Example 8.3. Integrate the geometric series

$$\frac{1}{1-x} = \sum_{n=0}^{\infty} x^n.$$

Solution

Integrate both sides.

$$\int \frac{1}{1-x}\,dx = -\ln(1-x) + C.$$

On the right,

$$\int \sum_{n=0}^{\infty} x^n\,dx = C + \sum_{n=0}^{\infty} \frac{x^{n+1}}{n+1}.$$

Thus,

$$\boxed{-\ln(1-x) = \sum_{n=0}^{\infty} \frac{x^{n+1}}{n+1}, \qquad |x| < 1,}$$

where the constant of integration is chosen so that both sides equal zero when $x = 0$.
Reindexing with $k = n + 1$,

$$\boxed{-\ln(1-x) = \sum_{k=1}^{\infty} \frac{x^k}{k}, \qquad |x| < 1.}$$

8.6 Example 3

Example 8.4. Given

$$f(x) = \sum_{n=0}^{\infty} \frac{x^n}{n!},$$

find a power series for

$$f'(x).$$

Solution

Differentiate term-by-term.

$$f'(x) = \sum_{n=1}^{\infty} \frac{n}{n!} x^{n-1}.$$

Since

$$\frac{n}{n!} = \frac{1}{(n-1)!},$$

we obtain

$$f'(x) = \sum_{n=1}^{\infty} \frac{x^{n-1}}{(n-1)!}.$$

Let

$$k = n - 1.$$

Then

$$\boxed{f'(x) = \sum_{k=0}^{\infty} \frac{x^k}{k!}.}$$

Thus,

$$\boxed{f'(x) = f(x).}$$

Later we will identify this function as

$$e^x.$$

8.7 Reindexing a Series

After differentiating or integrating, the index often changes.
For example,

$$\sum_{n=1}^{\infty} nx^{n-1}$$

may be rewritten by letting

$$k = n - 1.$$

Then

$$n = k + 1,$$

and

$$\boxed{\sum_{n=1}^{\infty} nx^{n-1} = \sum_{k=0}^{\infty} (k+1)x^k.}$$

Reindexing does not change the series; it simply makes the notation more convenient.

9. Taylor and Maclaurin Series

Power series provide an infinite polynomial representation of a function. The natural question is:

> Given a function $f(x)$, how can we determine the coefficients of its power series?

The answer is given by the **Taylor Series**. By repeatedly differentiating the function and evaluating the derivatives at a chosen point, we obtain a unique power series representation.
Taylor series are among the most important tools in mathematics because they allow us to approximate complicated functions using polynomials.

9.1 Taylor Polynomial

Suppose that $f(x)$ is sufficiently differentiable near a point $x = c$.
We seek a polynomial

$$P_n(x) = a_0 + a_1(x - c) + a_2(x - c)^2 + \cdots + a_n(x - c)^n$$

that approximates $f(x)$ near $x = c$.
This polynomial is called the **Taylor polynomial of degree** n.

9.2 Determining the Coefficients

We require

$$P_n(c) = f(c).$$

Therefore,

$$a_0 = f(c).$$

Differentiate.

$$P_n'(x) = a_1 + 2a_2(x - c) + \cdots .$$

Evaluating at

$$x = c,$$

gives

$$a_1 = f'(c).$$

Differentiate again.

$$P_n''(c) = 2a_2.$$

Hence,

$$a_2 = \frac{f''(c)}{2!}.$$

Continuing in this manner,

$$a_n = \frac{f^{(n)}(c)}{n!}.$$

9.3 Taylor Series

Substituting the coefficients into the polynomial produces the infinite series.

Theorem 9.1 (Taylor Series)**.** *If a function $f(x)$ is infinitely differentiable near $x = c$, then its Taylor series is*

$$\boxed{f(x) = \sum_{n=0}^{\infty} \frac{f^{(n)}(c)}{n!}(x-c)^n.}$$

Equivalently,

$$\boxed{f(x) = f(c) + f'(c)(x-c) + \frac{f''(c)}{2!}(x-c)^2 + \frac{f^{(3)}(c)}{3!}(x-c)^3 + \cdots.}$$

9.4 Maclaurin Series

When the center is

$$c = 0,$$

the Taylor series becomes the **Maclaurin Series**.

Definition 9.2. The Maclaurin series of

$$f(x)$$

is

$$\boxed{f(x) = \sum_{n=0}^{\infty} \frac{f^{(n)}(0)}{n!}x^n.}$$

Thus every Maclaurin series is a Taylor series centered at the origin.

9.5 Example 1: Maclaurin Series for e^x

Example 9.3. Find the Maclaurin series for

$$f(x) = e^x.$$

Solution
Since

$$f^{(n)}(x) = e^x$$

for every derivative,

$$f^{(n)}(0) = 1.$$

Therefore,

$$\boxed{e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!}.}$$

Expanding,

$$\boxed{e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} + \cdots .}$$

9.6 Example 2: Maclaurin Series for $\sin x$

Example 9.4. Find the Maclaurin series for

$$\sin x.$$

Solution
The derivatives repeat every four steps.

n	$f^{(n)}(x)$	$f^{(n)}(0)$
0	$\sin x$	0
1	$\cos x$	1
2	$-\sin x$	0
3	$-\cos x$	-1
4	$\sin x$	0

Only the odd powers remain.
Hence,

$$\boxed{\sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \cdots .}$$

or

$$\boxed{\sin x = \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n+1}}{(2n+1)!}.}$$

9.7 Example 3: Maclaurin Series for $\cos x$

Example 9.5. Find the Maclaurin series for

$$\cos x.$$

The derivatives are

$$\cos x, -\sin x, -\cos x, \sin x, \cos x, \dots$$

Evaluating at zero,

$$1, 0, -1, 0, 1, \dots$$

Therefore,

$$\boxed{\cos x = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \cdots.}$$

Equivalently,

$$\boxed{\cos x = \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n}}{(2n)!}.}$$

9.8 Example 4: Geometric Function

Recall that

$$\frac{1}{1-x} = 1 + x + x^2 + x^3 + \cdots, \qquad |x| < 1.$$

Thus,

$$\boxed{\frac{1}{1-x} = \sum_{n=0}^{\infty} x^n.}$$

This is one of the most important Maclaurin series because many other series can be obtained from it by differentiation or integration.

9.9 Example 5: Maclaurin Series for $\ln(1+x)$

Integrating the geometric series,

$$\frac{1}{1+x} = 1 - x + x^2 - x^3 + \cdots,$$

term-by-term gives

$$\boxed{\ln(1+x) = x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \cdots, \qquad -1 < x \le 1.}$$

Equivalently,

$$\boxed{\ln(1+x) = \sum_{n=1}^{\infty} (-1)^{n-1} \frac{x^n}{n}.}$$

9.10 Important Maclaurin Series

The following series should be memorized.

$$\boxed{\begin{aligned} e^x &= \sum_{n=0}^{\infty} \frac{x^n}{n!}, \\ \sin x &= \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n+1}}{(2n+1)!}, \\ \cos x &= \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n}}{(2n)!}, \\ \frac{1}{1-x} &= \sum_{n=0}^{\infty} x^n, \qquad |x| < 1, \\ \ln(1+x) &= \sum_{n=1}^{\infty} (-1)^{n-1} \frac{x^n}{n}, \qquad -1 < x \leq 1. \end{aligned}}$$

10. Taylor's Theorem and Error Estimation

A Taylor polynomial provides an approximation to a function near its center. In practice, however, we use only finitely many terms of the Taylor series.
Naturally, this raises the following question:

> How close is the Taylor polynomial to the actual function?

Taylor's Theorem answers this question by expressing the difference between the function and its Taylor polynomial as a remainder term.

10.1 Taylor Polynomial Revisited

Recall that the Taylor polynomial of degree n centered at $x = c$ is

$$P_n(x) = \sum_{k=0}^{n} \frac{f^{(k)}(c)}{k!}(x-c)^k.$$

This polynomial agrees with the function and its first n derivatives at the point

$$x = c.$$

10.2 The Error (Remainder)

The difference between the function and its Taylor polynomial is called the **remainder**.

$$\boxed{R_n(x) = f(x) - P_n(x).}$$

Therefore,

$$\boxed{f(x) = P_n(x) + R_n(x).}$$

The goal is to estimate the size of

$$R_n(x).$$

10.3 Taylor's Theorem (Lagrange Form)

Theorem 10.1 (Taylor's Theorem)**.** *Suppose*

$$f(x)$$

has continuous derivatives through order

$$n+1$$

on an interval containing

$$c$$

and

$$x.$$

Then

$$\boxed{R_n(x) = \frac{f^{(n+1)}(\xi)}{(n+1)!}(x-c)^{n+1},}$$

for some number

$$\xi$$

between

$$c$$

and

$$x.$$

Since the exact value of

$$\xi$$

is usually unknown, we estimate the remainder instead.

10.4 The Taylor Inequality

Suppose there exists a constant

$$M$$

such that

$$|f^{(n+1)}(x)| \leq M$$

for every

$$x$$

between

$$c$$

and the point of approximation.

Then

$$\boxed{|R_n(x)| \le \frac{M|x-c|^{n+1}}{(n+1)!}.}$$

This inequality is the principal tool for estimating approximation errors.

10.5 How to Estimate an Error

To estimate the error when using a Taylor polynomial:

1. Find the next derivative,
$$f^{(n+1)}(x).$$
2. Determine an upper bound
$$M$$
for its absolute value on the interval.
3. Substitute into
$$|R_n(x)| \le \frac{M|x-c|^{n+1}}{(n+1)!}.$$
4. Simplify the result.

10.6 Example 1

Example 10.2. Approximate

$$e^{0.2}$$

using the second-degree Maclaurin polynomial.
Estimate the maximum possible error.

Solution

Since

$$e^x = 1 + x + \frac{x^2}{2!} + \cdots,$$

the second-degree polynomial is

$$P_2(x) = 1 + x + \frac{x^2}{2}.$$

Evaluate at

$$x = 0.2.$$

$$P_2(0.2) = 1 + 0.2 + \frac{0.04}{2} = 1.22.$$

Now estimate the error.
The next derivative is

$$e^x.$$

On the interval

$$0 \le x \le 0.2,$$

we have

$$e^x \le e^{0.2}.$$

Choose

$$M = e^{0.2}.$$

Therefore,

$$|R_2(0.2)| \le \frac{e^{0.2}(0.2)^3}{3!}.$$

Since

$$3! = 6,$$

$$|R_2(0.2)| \le \frac{e^{0.2}(0.008)}{6}.$$

Thus,

$$\boxed{|R_2(0.2)| < 0.002.}$$

Therefore,

$$e^{0.2} \approx 1.22,$$

with an error less than approximately

$$0.002.$$

10.7 Example 2

Example 10.3. Approximate

$$\sin(0.3)$$

using the third-degree Maclaurin polynomial.

Solution

The third-degree polynomial is

$$P_3(x) = x - \frac{x^3}{6}.$$

Thus,

$$P_3(0.3) = 0.3 - \frac{0.027}{6} = 0.2955.$$

The next derivative is either

$$\sin x$$

or

$$\cos x,$$

both of which satisfy

$$|f^{(4)}(x)| \leq 1.$$

Hence,

$$M = 1.$$

Applying the Taylor Inequality,

$$|R_3(0.3)| \leq \frac{(0.3)^4}{4!} = \frac{0.0081}{24} \approx 3.38 \times 10^{-4}.$$

Therefore,

$$\boxed{\sin(0.3) \approx 0.2955,}$$

with an error less than

$$3.38 \times 10^{-4}.$$

10.8 Choosing the Degree of a Taylor Polynomial

Sometimes the desired accuracy is specified in advance.
For example,

> Approximate
>
> $$e^{0.5}$$
>
> with an error less than
>
> $$10^{-5}.$$

In such cases,

1. write the Taylor Inequality,
2. substitute the appropriate value of

$$M,$$

3. increase the degree

$$n$$

until

$$\frac{M|x-c|^{n+1}}{(n+1)!} < \text{required error.}$$

10.9 When Does the Taylor Series Equal the Function?

Having infinitely many derivatives is not, by itself, enough to ensure that the Taylor series equals the function.

A sufficient condition is that the remainder tends to zero:

$$\boxed{\lim_{n\to\infty} R_n(x) = 0.}$$

When this happens,

$$\boxed{f(x) = \sum_{n=0}^{\infty} \frac{f^{(n)}(c)}{n!}(x-c)^n.}$$

For the elementary functions studied in this course—such as

$$e^x,\ \sin x,\ \cos x,\ \ln(1+x),\ \frac{1}{1-x},$$

on their appropriate intervals—the Taylor series does equal the function.